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Name:

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Program: MSc

Semester: Autumn 2026

ALLAMA IQBAL OPEN UNIVERSITY, ISLAMABAD

Course: Mathematics for Economists (803) • Level: MSc

Semester: Autumn 2026

Assignment No. 2 (Units 6–9)

Total Marks: 100 • Passing Marks: 40 • Credit Hours: 03

SOLVED ASSIGNMENT GUIDE

(For Study & Reference)

Note: This guide is prepared for study and reference purposes. Students are advised to read the relevant units of the course book, understand the concepts and the working of each method, and then write the assignment in their own words.

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Question 1: Differentiating Exponential Functions (20 Marks)

Find dY/dX from the functions: (a) Y = e^(ax + b) (b) Y = x²e^x

Introduction

The exponential function e^x is the most important special function in mathematical economics, and this question tests the two standard techniques needed to differentiate expressions built from it. Part (a), Y = e^(ax + b), requires the chain rule, because the exponential is applied not to x alone but to the linear expression ax + b. Part (b), Y = x²e^x, requires the product rule, because the function is a product of a power of x and an exponential. Between them, the two parts therefore revise both of the great working rules of differentiation

in the setting where they are most often needed. This answer first gathers the background facts — what e is, why the derivative of e^x is itself, and the statements of the chain and product rules — then works each part step by step, checks each answer, and closes with the economic settings in which exactly these forms appear.

Background: The Function e^x and the Two Rules

The number e ≈ 2.71828 is the base of natural logarithms. It arises whenever growth is continuous: a sum of money growing at a continuously compounded interest rate, a population growing continuously, or a stock of capital depreciating continuously all follow exponential laws built on e. The defining property of the function e^x, and the reason for its central place in calculus, is that it is its own derivative:

d/dx (e^x) = e^x

No other function has this property, and nearly every formula in this answer flows from it. Two working rules are also needed. The chain rule handles a function of a function: if Y = f(u) where u = g(x), then dY/dx = (dY/du)(du/dx); applied to an exponential, it says that the derivative of e raised to any expression is the exponential itself multiplied by the derivative of the expression. The product rule handles a product of two functions: if Y = u(x) × v(x), then dY/dx = u (dv/dx) + v (du/dx) — in words, each factor is differentiated in turn while the other is held as it is, and the two results are added. A reliable memory of both rules, in both their forms, is assumed in what follows and is the real skill being examined.

Part (a): Y = e^(ax + b)

Set u = ax + b, so that Y = e^u. The chain has two links. The outer link gives dY/du = e^u, by the defining property just stated. The inner link gives du/dx = a, because the derivative of ax with respect to x is the constant a, and the derivative of the constant b is zero. Multiplying the links:

$$dY/dx = e^u \times a = a\,e^{(ax + b)}$$

So the answer to part (a) is dY/dx = a·e^(ax + b): the function reproduces itself, scaled by the constant a. Three remarks help to make the result stick. First, the constant b has disappeared from the multiplier but remains in the exponent — shifting an exponential curve sideways changes its level, not its rate of growth. Second, the special case a = 1, b = 0 returns d/dx(e^x) = e^x, so the general formula contains the basic fact as it should; checking a general answer against a known special case is a habit worth keeping. Third, the result can be read as a growth rate: dividing the derivative by the function itself gives (dY/dx)/Y = a, a constant. Functions of the form e^(ax + b) are exactly the functions that grow at a constant proportional rate a, which is why economists write continuous growth paths in this form — output growing at 3 per cent per year, for example, is Y = Y0 e^(0.03t).

Part (b): Y = x^2e^x

Here Y is the product of u = x^2 and v = e^x. Their derivatives are du/dx = 2x by the power rule, and dv/dx = e^x by the exponential property. Apply the product rule, differentiating each factor in turn:

$$dY/dx = u\,(dv/dx) + v\,(du/dx) = x^2\,(e^x) + e^x\,(2x)$$

Both terms contain the common factors x and e^x, so the answer should be factorised for neatness and for later use — a factorised derivative makes finding stationary points in optimisation problems far easier:

$$\mathbf{dY/dx = x\,e^{x}\,(x + 2)}$$

This is the answer to part (b). A spot check protects against slips: at x = 1 the formula gives dY/dx = 1 × e × 3 = 3e ≈ 8.15; estimating the slope of $Y = x^2e^x$ near x = 1 from values, Y(1) = e ≈ 2.718 and Y(1.1) = 1.21 × e^1.1 ≈ 1.21 × 3.004 ≈ 3.635, gives a slope of about (3.635 − 2.718)/0.1 ≈ 9.2 over the interval, which is consistent with a slope rising from 8.15 at x = 1 — the formula is behaving correctly. Notice also the pattern worth remembering: differentiating a polynomial times e^x returns another polynomial (of the same degree) times e^x, so repeated differentiation of such products never leaves this family of functions.

Why These Two Forms Matter in Economics

Forms (a) and (b) are not examination inventions; they recur across the subject. Exponential functions of the form in part (a) describe compound growth and decay: present-value discounting writes the value today of a future sum as A e^(−rt), depreciation writes a capital stock as K0 e^(−δt), and learning and diffusion curves take the same family of shapes, so differentiating them — to obtain the rate of decay of a present value, for instance — is a routine professional task. Products of powers and exponentials, as in part (b), appear in probability distributions used in econometrics (the gamma and chi-square densities are built from x raised to a power times e raised to a negative power of x), in production functions with exponential technical progress, Y = A e^(gt) K^α L^β, whose growth rate is found exactly by the

product and chain rules, and in present-value integrals where a polynomial payment stream is discounted by an exponential factor. A student fluent in these two differentiations is equipped for all of those applications.

It is also worth noting the close relative of part (a): the derivative of a general exponential a^x, or of the natural logarithm, is handled by the same chain-rule logic. In particular, logarithmic differentiation — taking logs of both sides, differentiating, and multiplying back — is the professional shortcut for products and powers like part (b) when they grow complicated, and it rests on nothing more than the chain rule applied to ln Y. Students who meet a product of three or more factors in later units will find that route calmer than applying the product rule repeatedly.

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Common Mistakes

The characteristic errors in this question are few and avoidable. In part (a), the commonest is to write dY/dx = e^(ax + b) and forget to multiply by the derivative of the exponent, a — the very factor the chain rule exists to supply. The second is to differentiate the exponent wrongly, treating b as though it contributed a term; b is a constant and vanishes. In part (b), the commonest error is to differentiate the product as though it were a product of derivatives, writing 2x × e^x alone and dropping the other product-rule term; the rule has two terms because either factor may be the one that changes. Writing u and v out explicitly before starting, as this answer does, is the simple discipline that prevents both mistakes.

Conclusion

In conclusion, the chain rule gives, for part (a), dY/dx = a e^(ax + b) — the exponential reproduces itself, multiplied by the derivative of its exponent — and the product rule gives, for part (b), dY/dx = x^2e^x + 2x e^x = x e^x (x + 2) in factorised form. Both results have been checked against special cases and numerical estimates, and both are instances of general rules that govern growth, discounting and distribution functions throughout economics.

Question 2: Second and Third Derivatives (20 Marks)

Find second and third derivatives of the following functions. (a) $y = 5x^5 - 4x + 13$ (b) $y = 11 - x^4$

Introduction

Differentiation does not have to stop after one step. The derivative of a function is itself a function, so it can be differentiated again to give the second derivative, and again to give the third, and so on for as long as the function allows. These higher-order derivatives are far from a formality: the second derivative is the single most used test in economic optimisation and in the study of curvature, as later questions in this assignment show, and higher derivatives underlie polynomial approximations of economic functions. Both functions in this question are polynomials, which makes them the ideal training ground: a polynomial is differentiated term by term by the power rule, each differentiation lowers every power by one, and after enough steps only a constant — and then zero — remains. This answer explains the notation and the meaning of the higher derivatives, works both functions through their first, second and third derivatives step by step, and checks the results.

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Notation and Meaning

If y = f(x), the first derivative is written dy/dx or f′(x); the second derivative — the derivative of the first derivative — is written d^2y/dx^2 or f″(x); the third is d^3y/dx^3 or f‴(x). The small raised numbers count how many times differentiation has been carried out; they are not powers of the derivative, a point of notation that beginners sometimes confuse. Economically, the first derivative is a rate of change: marginal cost, marginal product, marginal utility. The second derivative is the rate of change of that rate: it tells us whether the marginal magnitude is itself rising or falling. A falling marginal product of labour means the second derivative of the production function is negative; a U-shaped marginal cost curve turning upward means the second derivative of the cost function has become positive. The second derivative also measures curvature — concave when negative, convex when positive — and the third derivative, where it is needed, measures how that curvature itself is changing.

The working rule is the power rule applied repeatedly: the derivative of $a \cdot x^n$ is $a \cdot n \cdot x^{n-1}$, and the derivative of any constant term is zero. Nothing else is required for this question, but the work must be done in stages, differentiating the function that the previous stage produced — never returning to the original and multiplying things together. Writing each derivative on its own labelled line, as below, keeps the stages separate and makes the work easy to check and easy to mark.

Part (a): $y = 5x^5 - 4x + 13$

Differentiate term by term. The term $5x^5$ gives $5 \times 5x^4 = 25x^4$. The term $-4x$ gives -4. The constant 13 gives 0. Hence the first derivative is:

$$dy/dx = 25x^4 - 4$$

Now differentiate this result. The term $25x^4$ gives $25 \times 4x^3 = 100x^3$. The term -4 is now a constant, so it vanishes. Hence the second derivative is:

$$d^2y/dx^2 = 100x^3$$

Differentiate once more. The term $100x^3$ gives $100 \times 3x^2 = 300x^2$. Hence the third derivative is:

$$d^3y/dx^3 = 300x^2$$

So for part (a): the second derivative is $100x^3$ and the third derivative is $300x^2$. Notice how the pattern of the calculation can be summarised: each step multiplies the coefficient by the current power and reduces the power by one, 5 → 25 → 100 → 300 in the coefficients and 5 → 4 → 3 → 2 in the powers. Following the pattern in one's head while writing the lines out is a reliable internal check: if a coefficient fails to match the pattern, an arithmetic slip has occurred. Notice too the fate of the lower terms: $-4x$ survived only into the first derivative, and 13 did not survive at all, because constants and then linear terms are successively wiped out.

Part (b): $y = 11 - x^4$

Write the function as $y = 11 - x^4$ and again proceed in stages. The constant 11 gives 0; the term $-x^4$ gives $-4x^3$. Hence the first derivative is:

$$dy/dx = -4x^3$$

Differentiate this result. The term $-4x^3$ gives $-4 \times 3x^2 = -12x^2$. Hence the second derivative is:

$$d^2y/dx^2 = -12x^2$$

Differentiate once more. The term $-12x^2$ gives $-12 \times 2x = -24x$. Hence the third derivative is:

$$\mathbf{d^3y/dx^3 = -24x}$$

So for part (b): the second derivative is $-12x^2$ and the third derivative is $-24x$. The negative signs deserve attention rather than suspicion: the original term was $-x^4$, a downward-opening shape, and negativity in the leading term carries through every subsequent derivative. Students often lose marks by dropping a minus sign at the first step and then differentiating a wrong function flawlessly thereafter; keeping the sign attached to its term from the very first line is the safeguard. As a further check, the coefficient pattern should read $1 \rightarrow 4 \rightarrow 12 \rightarrow 24$ in magnitude as the powers fall $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$: each coefficient is the previous one multiplied by the power just used, exactly as the power rule requires.

Table 1: Summary of derivatives for both functions

Function	First derivative	Second derivative	Third derivative
(a) $y = 5x^5 - 4x + 13$	$25x^4 - 4$	$100x^3$	$300x^2$
(b) $y = 11 - x^4$	$-4x^3$	$-12x^2$	$-24x$

Reading the Results

Higher derivatives have content, not just form, and it is worth reading these results rather than merely producing them. For part (a), the second derivative $100x^3$ changes sign at $x = 0$: it is negative for $x < 0$ and positive for $x > 0$, so the curve is concave on the left of the origin and convex on the right, with a point of inflection at $x = 0$ — a preview of the convexity question

later in this assignment. For part (b), the second derivative − $12x^2$ is negative everywhere except at x = 0, where it is zero: the function 11 − x^4 is a downward hump with its peak at the origin, concave on both sides of the peak, exactly as the negative second derivative announces. In economics this kind of reading is used constantly: the sign of a second derivative classifies maxima and minima, tests whether a cost function is well behaved, and states the law of diminishing marginal returns in symbolic form.

One more general observation completes the picture. A polynomial of degree n can be differentiated n times before it becomes a constant, and n + 1 times before it vanishes altogether: part (a) is of degree 5, so its fourth and fifth derivatives are 600x and 600, and its sixth is zero; part (b) is of degree 4, so its fourth derivative is the constant − 24 and its fifth is zero. Non-polynomial functions behave differently — e^x, met in Question 1, never vanishes under differentiation, and 1/x never settles to a constant — which is one reason polynomials are used as approximations to more complicated functions: locally, a function's behaviour is summarised by its first few derivatives, a fact formalised in Taylor's theorem at a more advanced level.

Conclusion

In conclusion, repeated application of the power rule gives, for (a) $y = 5x^5 - 4x + 13$, the second derivative $d^2y/dx^2 = 100x^3$ and the third derivative $d^3y/dx^3 = 300x^2$; and for (b) $y = 11 - x^4$, the second derivative $d^2y/dx^2 = -12x^2$ and the third derivative $d^3y/dx^3 = -24x$. Each stage was obtained by differentiating the previous line only, the coefficient-and-power pattern checks

confirm the arithmetic, and the signs of the second derivatives correctly describe the curvature of the two curves.

Question 3: Relative Extrema of Two Functions (20 Marks)

Find the relative extremum of the function: (i) $Y = 5x^2 - x$ (ii) $Y = x^3 - 2x^2 + 3$

Introduction

Optimisation — finding the largest or smallest value of something — is the heart of economic theory. Firms maximise profit and minimise cost; consumers maximise utility; governments weigh the minimum point of an average-cost curve when regulating. The calculus tool for all such problems is the study of relative extrema: the peaks and troughs of a function relative to their immediate neighbourhood. This question asks for the relative extrema of two functions, a quadratic and a cubic, and the contrast between them is itself instructive: a quadratic has exactly one extremum, while a cubic may have two, one maximum and one minimum. This answer sets out the method — the first-derivative condition for finding candidate points and the second-derivative test for classifying them — applies it completely to both functions, verifies the results numerically and graphically, and explains the economic reading of the test.

The Method: First and Second Order Conditions

At a smooth peak or trough, the tangent to the curve is horizontal, so the slope — the first derivative — is zero. Setting the first derivative equal to zero and solving, the first-order (necessary) condition, therefore locates the stationary or critical points, the only places where a relative extremum can occur. But a zero slope does not by itself say whether the point is a peak,

a trough, or a point of inflection where the curve merely pauses while crossing its tangent, so a classification test is needed. The second-derivative test supplies it. If at a stationary point the second derivative is positive, the curve is bending upwards there and the point is a relative minimum; if the second derivative is negative, the curve is bending downwards and the point is a relative maximum; if the second derivative is zero, the test is inconclusive and the point must be examined by looking at the sign of the first derivative on either side. In the language of economics these are the first-order and second-order conditions for an optimum, and quoting both — rather than presenting a bare x-value — is what a complete answer consists of. A relative extremum, finally, means the extreme value compared with nearby points; it need not be the largest or smallest value of the function over its whole domain, a distinction that matters for the cubic below, which rises without bound as x grows.

Part (i): $Y = 5x^2 - x$

Differentiate: $Y' = 10x - 1$. Apply the first-order condition, setting the derivative to zero:

$$10x - 1 = 0 \quad \Rightarrow x = 1/10 = 0.1$$

There is exactly one stationary point, as expected for a quadratic. Now differentiate again: $Y'' = 10$. This is positive — indeed it is positive everywhere, not just at the stationary point, because the second derivative of a quadratic is a constant. By the second-derivative test, the stationary point is a relative minimum. It remains to report the minimum value of the function, which many candidates forget: an extremum is a value of Y, located at a value of x, and both should be stated. Substituting $x = 0.1$:

$$Y = 5\,(0.1)^2 - 0.1 = 5\,(0.01) - 0.1 = 0.05 - 0.1 = -0.05 = -1/20$$

So the function $Y = 5x^2 - x$ has a single relative extremum: a relative minimum of $-1/20$ (-0.05), attained at $x = 1/10$ (0.1). The result is geometrically obvious once the function is recognised as an upward-opening parabola — its coefficient of x^2 is positive — since such a parabola has a single lowest point, its vertex, and no maximum. It is also the same function that returns in Question 4 of this assignment, where its constant positive second derivative will be the proof that it is convex everywhere; the student should note how the ideas of this course interlock.

Part (ii): $Y = x^3 - 2x^2 + 3$

Differentiate: $Y' = 3x^2 - 4x$. Apply the first-order condition and solve the resulting quadratic, factorising:

$$3x^2 - 4x = 0 \;\Rightarrow\; x(3x - 4) = 0 \;\Rightarrow\; x = 0 \text{ or } x = 4/3$$

There are two stationary points, and each must be classified separately. Differentiate again: $Y'' = 6x - 4$. Test the first point, $x = 0$: $Y''(0) = -4$, which is negative, so $x = 0$ is a relative maximum. Its value is $Y = 0 - 0 + 3 = 3$. Test the second point, $x = 4/3$: $Y''(4/3) = 6 \times (4/3) - 4 = 8 - 4 = 4$, which is positive, so $x = 4/3$ is a relative minimum. Its value must be computed with care, using fractions rather than decimals to keep it exact: $(4/3)^3 = 64/27$; $2(4/3)^2 = 2 \times 16/9 = 32/9 = 96/27$; and the constant $3 = 81/27$. Therefore $Y = 64/27 - 96/27 + 81/27 = 49/27 \approx 1.81$.

So the function $Y = x^3 - 2x^2 + 3$ has two relative extrema: a relative maximum of 3 at $x = 0$, and a relative minimum of $49/27$ (≈ 1.81) at $x = 4/3$ (≈ 1.33). The shape this describes is the characteristic cubic wave: the curve climbs to its peak at the origin, descends into its trough at $x = 4/3$, and then climbs forever. Note that the maximum value (3) exceeds the minimum value

(49/27), as consistency demands, and that neither extremum is a global one: as $x \to -\infty$ the function falls without limit, and as $x \to +\infty$ it rises without limit, which is exactly why the question correctly asks for relative, not absolute, extrema.

Table 1: Summary of stationary points and their classification

Function	Stationary point	Second derivative there	Classification	Extreme value
(i) $Y = 5x^2 - x$	$x = 0.1$	$Y'' = 10 > 0$	Relative minimum	$Y = -0.05$
(ii) $Y = x^3 - 2x^2 + 3$	$x = 0$	$Y'' = -4 < 0$	Relative maximum	$Y = 3$
(ii) $Y = x^3 - 2x^2 + 3$	$x = 4/3$	$Y'' = 4 > 0$	Relative minimum	$Y = 49/27 \approx 1.81$

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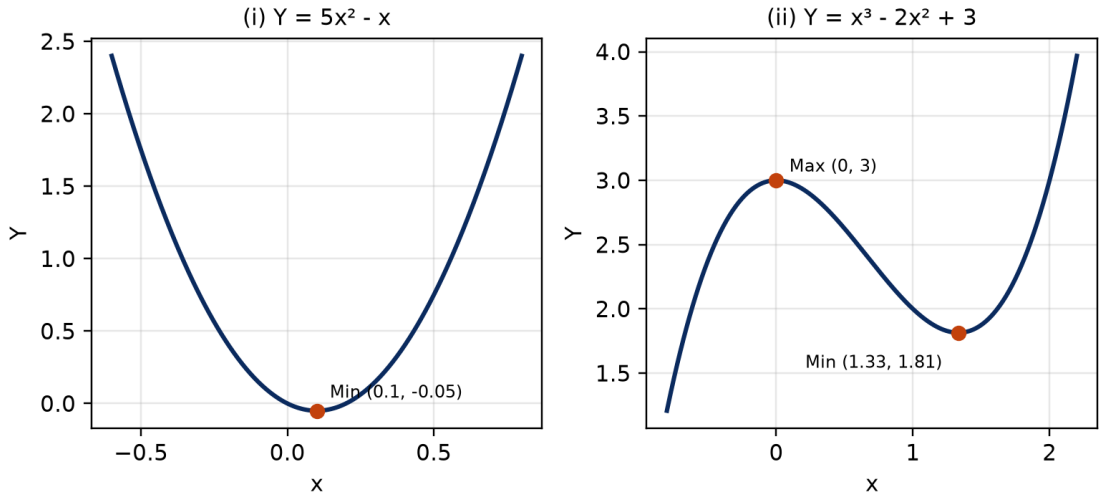


Figure 1: The extrema found in this question. Left: the parabola $Y = 5x^2 - x$ bottoms out at (0.1, − 0.05). Right: the cubic $Y = x^3 - 2x^2 + 3$ peaks at (0, 3) and dips at (4/3, 49/27).

Numerical Verification

Formulas should be checked against the function itself, and the sign pattern of the first derivative around each stationary point gives an independent confirmation. For part (ii), evaluate Y at points near the maximum: Y(− 0.5) = − 0.125 − 0.5 + 3 = 2.375, and Y(0.5) = 0.125 − 0.5 + 3 = 2.625 — both below Y(0) = 3, as a maximum requires. Near the minimum, Y(1) = 1 − 2 + 3 = 2 and Y(2) = 8 − 8 + 3 = 3 — both above 49/27 ≈ 1.81, as a minimum requires. For part (i), Y(0) = 0 and Y(0.2) = 5(0.04) − 0.2 = 0 — both above Y(0.1) = − 0.05, symmetric around the vertex exactly as a parabola demands. The derivative-sign story agrees too: for the cubic, Y′ = x(3x − 4) is positive before x = 0 (rising to the maximum), negative between 0 and 4/3 (falling to the minimum), and positive after 4/3 (rising away) — rise, fall, rise — which is the fingerprint of a maximum followed by a minimum. Whenever the second-derivative test is unavailable or inconclusive, this first-derivative sign check is the fallback method, and it is good practice to know it even when it is not needed.

Economic Significance

The procedure just practised is, line for line, the derivation of a firm's supply behaviour and a consumer's demand behaviour. A profit function π(Q) is maximised where its first derivative — marginal profit, that is marginal revenue minus marginal cost — is zero, and where its second derivative is negative; the famous rule MC = MR is nothing but the first-order condition of this question wearing economic clothes, and the requirement that marginal cost cut marginal revenue from below is the second-order condition. Likewise, the minimum point of an average-cost curve, found by exactly the steps used in part (i), defines the long-run equilibrium scale of

a competitive firm. Mastering extrema on clean polynomial examples is therefore direct preparation for the optimisation chapters of microeconomic theory.

Conclusion

In conclusion: the function $Y = 5x^2 - x$ has one relative extremum, a minimum value of − 1/20 at $x = 1/10$, certified by $Y'' = 10 > 0$. The function $Y = x^3 - 2x^2 + 3$ has two: a relative maximum value of 3 at $x = 0$, where $Y'' = -4 < 0$, and a relative minimum value of 49/27 at $x = 4/3$, where $Y'' = 4 > 0$. Numerical sampling around every point, the sign pattern of the first derivative, and the plotted curves all confirm the classification.

Question 4: Which Functions Are Convex? (20 Marks)

Which functions are convex (i) $x^3 - 2x^2 + 3$ (ii) $5x^2 - x$

Introduction

Convexity is the curvature property on which a large part of optimisation theory rests. A convex function has the priceless feature that any local minimum is automatically the global minimum, so that the first-order condition, which merely locates stationary points, becomes sufficient for an optimum. Economists therefore test for convexity constantly — of cost functions, of production relationships, of the sets over which consumers and firms choose. This question gives two functions, already familiar from Question 3 of this assignment, and asks which of them are convex. The honest answer requires care, because one of them is convex everywhere while the other changes its curvature part-way along its domain, and saying so precisely — with the intervals stated — is the substance of a good answer. This response defines convexity in

three equivalent ways, develops the second-derivative test, and then applies it to each function in turn.

What Convexity Means

Geometrically, a function is convex if the straight line segment (the chord) joining any two points on its graph lies on or above the graph between those points — the curve sags below its chords, like a bowl or the letter U. Algebraically, a function f is convex on an interval if for any two points x1 and x2 in the interval and any weight λ between 0 and 1:

$$f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2)$$

That is, the value of the function at an average of two points is no greater than the average of the values there — the function of the average is at most the average of the function. If the inequality is strict whenever $x_1 \neq x_2$ and $0 < \lambda < 1$, the function is strictly convex. For a twice-differentiable function of one variable there is a third, working characterisation, and it is the one used in practice: the function is convex exactly where its second derivative is non-negative, and strictly convex where its second derivative is positive. The reason is intuitive — a positive second derivative means the slope is steadily increasing, so the curve continually bends upward, which is precisely the bowl shape. A function with a negative second derivative throughout is concave instead (shaped like a cap), and a function whose second derivative changes sign is neither convex nor concave over the whole domain, though it may be each on different intervals. Note that convexity is always a claim about a function on a stated domain: responsible answers name the domain, and this question is an object lesson in why.

Function (ii) First: $f(x) = 5x^2 - x$

Take the simpler and better-behaved function first. Differentiate twice: $f'(x) = 10x - 1$, and $f''(x) = 10$. The second derivative is the constant 10, which is positive for every value of x without exception. By the second-derivative test, therefore, $f(x) = 5x^2 - x$ is strictly convex on the whole real line. This verdict matches everything else known about the function: it is a parabola opening upwards, its chord between any two points lies above the curve — the right-hand panel of the figure draws one such chord — and in Question 3 it possessed a single stationary point which was automatically its global minimum, the signature behaviour of a strictly convex function. Indeed every quadratic with a positive coefficient on x^2 is strictly convex everywhere, and every quadratic with a negative coefficient is strictly concave everywhere; quadratics never change curvature, because their second derivative is a constant.

Function (i): $f(x) = x^3 - 2x^2 + 3$

Now differentiate the cubic twice: $f'(x) = 3x^2 - 4x$, and $f''(x) = 6x - 4$. Unlike the previous case, this second derivative is not a constant; it is a line in x, and it changes sign where it passes through zero:

$$6x - 4 = 0 \quad \Rightarrow x = 2/3$$

For $x > 2/3$, $f''(x) > 0$, so the function is (strictly) convex on that range. For $x < 2/3$, $f''(x) < 0$, so the function is (strictly) concave on that range. The point $x = 2/3$, at which the curvature flips, is a point of inflection, and its height is $f(2/3) = 8/27 - 8/9 + 3 = 8/27 - 24/27 + 81/27 = 65/27 \approx 2.41$. The verdict must therefore be stated with the domain attached: the cubic is NOT convex on the whole real line — indeed it is not convex on any interval that extends to the left of 2/3

— but it IS strictly convex on the interval [2/3, ∞) and strictly concave on (− ∞, 2/3]. A reader can confirm the failure of global convexity with a single chord: between x = − 1 (where f = 0) and x = 1 (where f = 2), the chord's midpoint value at x = 0 is the average, 1, while the function itself gives f(0) = 3, which lies above the chord — the defining inequality is violated, so the function is not convex overall. By contrast, between x = 1 and x = 2, both in the convex region, the midpoint value f(1.5) = 3.375 − 4.5 + 3 = 1.875 lies below the chord average (2 + 3)/2 = 2.5, as convexity requires.

Table 1: Second-derivative sign and curvature of the two functions

Function	Second derivative	Sign	Verdict
(i) $x^3 - 2x^2 + 3$	$6x - 4$	Negative for $x < 2/3$; positive for $x > 2/3$	Not convex on the whole line: concave before, convex after, inflection at $x = 2/3$
(ii) $5x^2 - x$	10	Positive for all x	Strictly convex everywhere

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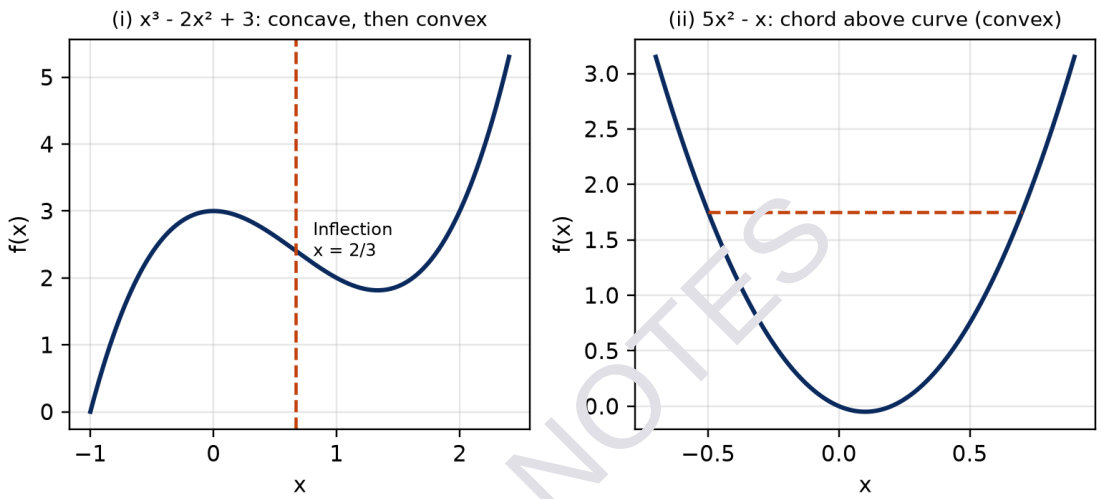


Figure 1: Left: the cubic bends from concave (cup-shaped) to convex (bowl-shaped) at its inflection point x = 2/3. Right: the parabola 5x² − x is convex everywhere, so the dashed chord lies above the curve.

Why Economists Care

Curvature assumptions are the working hypotheses of optimisation. A production function is usually assumed concave — diminishing marginal product — which makes the firm's profit function well behaved and its first-order conditions sufficient for a maximum. A total cost function that is convex in output — rising marginal cost — guarantees that the output where price equals marginal cost really is the profit maximum and indeed the unique one. Preferences are modelled by assuming utility functions concave (equivalently, preference sets convex), which is what makes a consumer's tangency point a genuine optimum rather than a minimum of satisfaction. When a function changes curvature, as the cubic here does, none of these guarantees survive globally: the same first-order condition produced, in Question 3, one maximum and one minimum for this very function, which could not happen if it were convex or concave throughout. The question's two functions, taken together, thus teach the essential

lesson: convexity is a property to be verified from the second derivative over the domain of interest, never a property to be assumed from a function's general appearance.

A final clarification guards against a common confusion. Convexity of a function, tested by f″ ≥ 0, should not be confused with quasi-concavity, the weaker property examined in Question 5, which concerns the shape of a function's level sets rather than the curvature of the function itself. Every concave function is quasi-concave, but the converse is false; likewise every convex function is quasi-convex, but not conversely. Keeping the family of curvature concepts separate — convex, concave, quasi-convex, quasi-concave — and knowing which test belongs to which, is part of the professional vocabulary this unit is building.

Conclusion

In conclusion: function (ii), $5x^2 - x$, has second derivative identically equal to $10 > 0$ and is therefore strictly convex on the whole real line. Function (i), $x^3 - 2x^2 + 3$, has second derivative $6x - 4$, which is negative for $x < 2/3$ and positive for $x > 2/3$; it is consequently not convex over its whole domain, being strictly concave on $(-\infty, 2/3)$, strictly convex on $(2/3, \infty)$, with a point of inflection at $(2/3, 65/27)$. So, of the two functions named in the question, only $5x^2 - x$ is convex (everywhere); the cubic is convex only on the restricted domain $x \geq 2/3$.

Question 5: Short Notes — Partial and Total Derivatives; Quasi-Concavity (10 + 10 Marks)

Write a short note on the following? • Partial and Total derivative. • Quasi Concave.

Introduction

This question asks for two short notes on concepts that complete the differentiation unit of the course. The first note, on partial and total derivatives, deals with functions of several variables and with the question of how to measure change when more than one thing is moving at once — a distinction already used in Assignment No. 1, Question 5, where a total derivative was calculated. The second note, on quasi-concavity, introduces the curvature property that economic theory most often assumes of utility and production functions, because it is exactly the property that guarantees sensible, convex sets of preferred or feasible choices. Each note defines its concept, explains it with a worked example, states its uses, and draws the contrasts that examiners look for.

Note (a): Partial and Total Derivatives

A partial derivative measures the rate of change of a function of several variables with respect to one of its variables, while all the other variables are held constant. If $z = f(x, y)$, the partial derivative with respect to x, written $\partial z/\partial x$ or f_x, is found by differentiating f with respect to x treating y as a constant; likewise $\partial z/\partial y$ treats x as constant. The curved symbol ∂ is the permanent reminder that other influences exist and have been frozen for the purpose of the calculation. The rule of working is therefore ordinary differentiation with a discipline attached: decide which variable is moving, freeze the rest, and differentiate as usual. For example, if $z = 3x^2y + y^3$, then $\partial z/\partial x = 6xy$ (y frozen, so 3y is a constant coefficient, and y^3 is a pure constant and vanishes), while $\partial z/\partial y = 3x^2 + 3y^2$ (x frozen, so the first term contributes its coefficient $3x^2$ and the second term differentiates normally). Partial derivatives are the mathematics of ceteris

paribus: the partial derivative of a demand function with respect to own price, holding income and other prices constant, is precisely the slope of the ordinary demand curve, and a marginal product is the partial derivative of the production function with respect to one input, holding the others fixed.

A total derivative, by contrast, measures the complete change in a variable when nothing is frozen — when every channel through which the change can travel is allowed to operate. Suppose z = f(x, y) and both x and y depend on a single underlying variable t, say x = g(t) and y = h(t). Then a change in t moves z directly through x and also through y, and the total derivative adds the two chain-rule contributions:

$$dz/dt = (\partial z/\partial x)(dx/dt) + (\partial z/\partial y)(dy/dt)$$

If t also appears in f itself, a direct term ∂z/∂t is added as a third contribution. The companion statement for small changes is the total differential, dz = (∂z/∂x) dx + (∂z/∂y) dy, which approximates the total change in z by adding the pieces contributed by each variable's change. A compact worked example ties the ideas together. Let $z = x^2y$, with $x = 2t$ and $y = t^2$. The partial derivatives are $\partial z/\partial x = 2xy$ and $\partial z/\partial y = x^2$; the inner derivatives are $dx/dt = 2$ and $dy/dt = 2t$. Hence $dz/dt = 2xy \times 2 + x^2 \times 2t = 4xy + 2t\,x^2$. Substituting $x = 2t$ and $y = t^2$: $dz/dt = 4(2t)(t^2) + 2t(4t^2) = 8t^3 + 8t^3 = 16t^3$. The check by direct substitution is immediate: $z = (2t)^2 (t^2) = 4t^4$, whose ordinary derivative is $16t^3$ — the same result, as it must be.

Table 1: Partial derivative versus total derivative

Aspect	Partial derivative	Total derivative

Symbol	Curved ∂, e.g. ∂z/∂x	Straight d, e.g. dz/dt
What moves	One variable only	All channels allowed to move
Other variables	Held constant (ceteris paribus)	Change according to their own functions
Economic meaning	A marginal magnitude: marginal product, slope of a demand curve	Full effect of a change: comparative statics, total revenue effect

The uses of the distinction run through the whole subject. Comparative statics — finding how an equilibrium price or income responds to a shift in a parameter — is performed by totally differentiating the equilibrium conditions, because in a new equilibrium everything adjusts together. The total differential is the tool for error-and-approximation calculations and for deriving results such as Euler's theorem on homogeneous functions. Confusing the two concepts, by reporting a partial effect as though it were the whole effect, is a substantive economic error and not a mere slip of notation: the total effect of a price change on demand, for instance, combines a substitution effect and an income effect travelling through different channels, and policy based on only one of them can be badly wrong. (This is also why the total derivative computed in Assignment No. 1, Question 5, contained both a chain-rule term, $4x(6t^2 + 3)$, and a direct term, $-3t^2$.)

Note (b): Quasi-Concave Functions

A function f is called quasi-concave if its upper level sets are convex sets: that is, for every value k, the set of points x for which f(x) ≥ k forms a convex set. An equivalent definition, often easier to apply, is that for any two points x and y and any weight λ between 0 and 1:

$$f(\lambda x + (1 - \lambda) y) \geq \min \{ f(x), f(y) \}$$

In words, the value of the function at any mixture of two points is at least as large as the smaller of the two values there: averaging two bundles never does worse than the worse of the two. For a consumer, that is exactly the assumption that balanced bundles are at least as good as extreme ones — the preference for variety — and it is what makes indifference curves bow inwards towards the origin, so that the set of bundles at least as good as a given bundle is a convex set. Quasi-concavity is weaker than concavity: concavity requires f at the mixture to be at least the average of the values, a stronger demand than being at least the minimum. Consequently every concave function is quasi-concave, but a quasi-concave function need not be concave. Two standard examples make the gap concrete. The function $f(x) = x^3$ is increasing, so each of its upper level sets is an interval of the form $[c, \infty)$, which is a convex set, and the function is quasi-concave — yet its second derivative $6x$ changes sign, so it is not concave on the whole line. Likewise the utility function $u = x \cdot y$ is quasi-concave — its indifference curves are convex to the origin — although, its exponents summing to more than one, it is not a concave function. What shapes choice, these examples show, is the pattern of the level sets, not the numerical curvature of the function. Moreover, any increasing transformation of a quasi-concave function remains quasi-concave, whereas concavity can be destroyed by such a

transformation. This last property is decisive in consumer theory, where utility numbers are only ordinal — meaningful up to increasing transformations — so any assumption placed on a utility function must be one that survives re-scaling of the numbers, and quasi-concavity passes that test while concavity does not.

For twice-differentiable functions of two variables, quasi-concavity can be tested with the bordered Hessian: for a function of two variables the relevant test is that the determinant of the 3×3 bordered Hessian matrix — bordered by the first derivatives — is non-negative, a test taught alongside the ordinary Hessian tests for concavity and convexity. In one variable the idea simplifies greatly: a function of one variable is quasi-concave if it first increases and then decreases — single-peaked — so that every upper level set is an interval; thus the cubic of Questions 3 and 4, restricted suitably, and ordinary hump-shaped functions such as $11 - x^4$ of Question 2, are quasi-concave on appropriate domains even where they fail to be concave everywhere. The mirror-image concept is quasi-convexity, defined by convex lower level sets, f at a mixture being no more than the maximum of the two values; cost and distance-type functions often carry that property.

Quasi-concavity is assumed at exactly the points of economic theory where well-behaved choices are needed. Utility functions are assumed quasi-concave so that indifference curves are convex to the origin, the marginal rate of substitution diminishes, and a budget-line tangency is a genuine maximum. Production functions are assumed quasi-concave so that isoquants are convex to the origin, the marginal rate of technical substitution diminishes, and input combinations average sensibly. In optimisation with inequality constraints, quasi-concavity of the objective together with convexity of the constraint set is what supports the sufficiency

results used in advanced work: a stationary point satisfying the Kuhn–Tucker conditions is then a global maximum. The concept is thus not a curiosity but the minimal curvature assumption under which the standard diagrams and theorems of microeconomics are valid.

Conclusion

To conclude the two notes: a partial derivative isolates one channel of change, holding all other variables constant, and is denoted by ∂; a total derivative sums the contributions of every channel, direct and indirect, when the variables move together, and is denoted by d — the first being the language of marginal analysis and the second the language of comparative statics. A quasi-concave function is one whose upper level sets are convex, equivalently one for which a mixture of two points scores at least the minimum of the two separate scores; it is weaker than concavity, survives increasing transformations, and is the standard curvature assumption on utility and production functions throughout economic theory.

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