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ALLAMA IQBAL OPEN UNIVERSITY, ISLAMABAD

Course: Mathematics for Economists (803) • Level: MSc

Semester: Autumn 2026

Assignment No. 1 (Units 1-5)

Total Marks: 100 • Passing Marks: 40 • Credit Hours: 03

SOLVED ASSIGNMENT GUIDE

(For Study & Reference)

Note: This guide is prepared for study and reference purposes. Students are advised to read the relevant units of the course book, understand the concepts and the working of each method, and then write the assignment in their own words.

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Question 1: The Mathematical Model; Endogenous and Exogenous Variables (20 Marks)

Explain the mathematical model. What are endogenous variables and exogenous variables?

Introduction

Economics studies relationships: how price affects quantity demanded, how income affects consumption, how the interest rate affects investment, and so on. Stated in ordinary words, such relationships are often long, vague and difficult to test. A mathematical model restates the same relationships briefly, exactly and in symbols, so that conclusions can be derived by logic rather than guessed by intuition. Because the MSc course Mathematics for Economists begins

with models, equilibrium and variables, this question is the foundation of everything that follows in the course: derivatives, optimisation and comparative statics are all, in the end, operations performed on a model. This answer first explains what a model in general is, then what a mathematical (economic) model is, lists its ingredients and the steps by which it is built, and finally defines and distinguishes endogenous and exogenous variables, with examples drawn from familiar economic models.

What Is a Model?

A model is a simplified representation of reality. It is not reality itself, and it is not meant to be: a road map is a model of a country, and it is useful precisely because it leaves out houses, trees and fields and keeps only roads and towns. In the same way, an economic model leaves out the countless details of real economic life — differences of taste, weather, politics, habit and chance — and keeps only the few factors that are believed to matter most for the problem in hand. If a model reproduced reality in every detail it would be as complicated as reality and therefore useless. The art of model building is the art of sensible simplification: abstract enough to be manageable, yet realistic enough that its conclusions still teach us something true about the world.

A mathematical model is a model in which the relationships among the chosen factors are written as equations (or inequalities) in symbols. Instead of writing, in a full sentence, that the quantity of a good which people wish to buy depends on its price, their income and the prices of related goods, the economist writes a demand function, Qd = f(P, Y, Pr), which says exactly the same thing in one line and, unlike the sentence, can be manipulated: it can be solved,

differentiated, graphed and combined with other equations. Mathematics therefore acts as a precise language and as a reasoning tool at the same time.

Ingredients of a Mathematical Economic Model

Every mathematical model in economics is built from the same small set of ingredients. Understanding these ingredients is the surest way to understand any model one meets later in the course.

- Variables. A variable is a magnitude that can take different values — price, quantity, income, output, cost and profit are variables. A variable is represented by a symbol such as P, Q, Y, C or r, and the work of solving a model is the work of finding the values of some of its variables.
- Constants and parameters. A constant is a magnitude that does not change within the model. A coefficient attached to a variable, such as the b in C = a + bY, is called a parameter: it is treated as given while the model is solved, although a different value of it would describe a different situation. Parameters often summarise influences that the model does not explain in detail.
- Equations. The relationships of the model are stated as equations, which are normally of three kinds. Definitional equations hold true by definition, for example profit equals total revenue minus total cost, $\pi = TR - TC$. Behavioural equations describe how agents are assumed to behave, for example the consumption function $C = a + bY$, which says households spend a fixed amount plus a fraction of income. Equilibrium (or condition)

equations state the condition that must hold when the system is at rest, for example Qd = Qs in a market, or saving equals investment in a simple income model.

- Assumptions. Every model rests on assumptions — about rationality, about competition, about what is held constant (the famous ceteris paribus clause, meaning other things equal). Assumptions should always be stated, because the conclusions of a model are only as reliable as the assumptions behind them.

How a Model Is Built and Used

Model building usually proceeds in four steps. First, the problem is defined and the important variables are selected. Second, assumptions are made and the relationships are written as equations; at this stage economic theory guides the builder, because theory tells us which variables belong in each equation and what sign each relationship is expected to have. Third, the model is solved: the equations are manipulated mathematically to obtain values, or general expressions, for the variables the model is meant to explain. Fourth, the solution is interpreted and, where data exist, tested. If the model's predictions repeatedly fail against real observations, the assumptions are revised and the model is rebuilt. A model is therefore never finished once and for all; it is a working tool that improves as evidence accumulates.

A small example makes the whole process concrete. Consider the market for a single good. Theory suggests the behavioural equations Qd = 100 − 2P (demand falls as price rises) and Qs = − 20 + 3P (supply rises as price rises), together with the equilibrium condition Qd = Qs. Solving, 100 − 2P = − 20 + 3P gives 120 = 5P, so P = 24, and substituting back gives Q = 100 − 48 = 52. From three short equations the model has produced a definite prediction: price 24 and quantity

52. It has also produced insight, because we can now ask what happens if demand rises — say the demand equation becomes Qd = 130 − 2P — and solve again to find the new price and quantity. This asking of what-if questions, called comparative statics, is one of the chief uses of models in economics.

Endogenous Variables

An endogenous variable is a variable whose value is determined from within the model — it is explained by the model itself. The word comes from Greek roots meaning generated from inside. In the market example just given, price P and quantity Q are endogenous: nobody fixes them from outside; they emerge from the interaction of demand and supply inside the model. In the simple Keynesian income model Y = C + I + G with C = a + bY, national income Y and consumption C are endogenous, because once the equations are solved their values follow from the model. The test for an endogenous variable is simple: if solving the model gives you its value, it is endogenous. The number of endogenous variables in a well-built model normally equals the number of independent equations, so that the system can in principle be solved — a model with more unknowns than equations cannot determine its own variables.

Exogenous Variables

An exogenous variable is a variable whose value is taken as given from outside the model — it influences the system, but the model does not try to explain it. The word means generated from outside. In the Keynesian model, investment I and government spending G are commonly treated as exogenous: they affect income, but this small model contains no equation explaining what determines them. In a demand function Qd = f(P, Y, Pr), if the model is only about one

market, the consumer's income Y and the price of a related good Pr are exogenous — they are data handed to the model. In the market example of this answer, the numbers 100 and 20 inside the demand and supply equations reflect such outside influences as tastes, population and input costs, all summarised in the intercepts rather than explained. When an exogenous variable changes, the whole solution of the model shifts, which is exactly why exogenous variables are sometimes called the model's policy variables or shift variables: government spending, taxes and the money supply are exogenous levers through which policy moves the endogenous outcomes.

It is essential to see that the distinction is relative to the model, not a permanent label on the variable itself. Price is endogenous in a market model but may be exogenous to a single small firm in perfect competition, which takes the market price as given. Investment is exogenous in the simplest income model but becomes endogenous in a larger model that adds an investment function I = I0 − dr depending on the interest rate. Enlarging a model typically means endogenising some variables that were previously treated as given, at the cost of more equations and more complexity.

Table 1: Endogenous and exogenous variables in two familiar models

Model	Endogenous (solved by the model)	Exogenous (given from outside)
Single market	Price P; quantity traded Q	Income, tastes, input costs (inside the intercepts and slopes)

Keynesian income model	Income Y; consumption C	Investment I; government spending G; parameters a and b

Why the Distinction Matters

The distinction is not a mere vocabulary exercise. First, it tells the student what a model can and cannot answer: a model cannot explain its exogenous variables, and criticising it for failing to do so misses the point of the simplification. Second, it organises comparative-static analysis, which is the comparison of an old equilibrium with a new one after an exogenous change — most of applied economic analysis, from tax changes to exchange-rate shocks, takes exactly this form. Third, in econometrics the distinction decides estimation strategy, because treating an in-fact-endogenous variable as exogenous is a classic source of biased results. Finally, it clarifies policy: policy instruments must be exogenous, or at least controllable, if they are to be used to steer the endogenous variables that society cares about, such as income, employment and the price level.

Conclusion

To conclude, a mathematical model is a simplified, symbolic statement of economic relationships, built from variables, parameters and equations — definitional, behavioural and equilibrium — resting on explicit assumptions, and valued because it can be solved, tested and used for what-if analysis. Within such a model, endogenous variables are those the model itself determines, while exogenous variables are those taken as given from outside, whose changes shift the solution. The same magnitude may be endogenous in one model and exogenous in

another, so the classification always depends on the model in hand. Keeping the two apart is the first step towards reading, solving and criticising every economic model correctly.

Question 2: Equilibrium; Partial and General Market Equilibrium (20 Marks)

Define equilibrium. Also, compare the partial market equilibrium and the general market equilibrium.

Introduction

Few words in economics carry more weight than equilibrium. Prices, output, employment and interest rates are all analysed as equilibrium outcomes, and a change in any economic situation is usually described as a movement from one equilibrium to another. Yet the word is borrowed from physics, and its economic meaning needs care. This answer first defines equilibrium, explains its main features and the related ideas of stability and disequilibrium, and then draws the comparison the question asks for: between partial market equilibrium, which studies one market in isolation in the tradition of Alfred Marshall, and general market equilibrium, which studies all markets together in the tradition of Léon Walras. A figure and a comparison table support the discussion, and a short numerical illustration shows partial equilibrium at work.

Meaning and Definition of Equilibrium

Equilibrium means a state of balance or rest — a position in which the opposing forces at work are equal, so that there is no tendency to change. An economic equilibrium is a situation in

which every agent's plan is consistent with every other agent's plan, so that, unless some outside factor changes, nobody has a reason to alter his behaviour. A consumer in equilibrium is maximising satisfaction given his budget; a firm in equilibrium is maximising profit given its costs and the market price; a market is in equilibrium when the quantity buyers wish to buy exactly equals the quantity sellers wish to sell, at the ruling price. Equilibrium can therefore be defined as follows: it is a state in which the economic forces of demand and supply (or, more generally, all the plans of decision-makers) are in balance, the plans are mutually fulfilled, and there exists no inherent tendency for the variables of the system to change.

Several features of this definition deserve emphasis. First, equilibrium does not mean that everything is still in a literal sense — trade, production and consumption all continue — it means that the pattern repeats itself because plans are being fulfilled. Second, equilibrium does not mean a good or desirable state: an economy can rest in an equilibrium with heavy unemployment, as Keynes emphasised, because equilibrium only means an absence of tendency to change, not the presence of welfare. Third, equilibrium is always defined relative to a set of given conditions — tastes, technology, incomes, the number of firms, and policy. Change any of these data and the old equilibrium is disturbed and the system moves towards a new one. Fourth, economists distinguish stable from unstable equilibrium: an equilibrium is stable if, after a small disturbance, the system returns to it, as a ball returns to the bottom of a bowl; it is unstable if a small disturbance pushes the system further away, as a ball balanced on top of a dome rolls off.

Disequilibrium and Adjustment

The opposite state, disequilibrium, exists when plans do not match: at a price above the market equilibrium there is excess supply (a surplus), unsold stocks accumulate, and sellers cut price; at a price below it there is excess demand (a shortage), queues form, and buyers bid the price up. In each case the imbalance itself generates the pressure that removes it, which is why the market equilibrium in a competitive market is normally stable. Studying the adjustment path matters in practice because real economies spend much of their time adjusting, and policies such as price controls can block the adjustment and freeze a market in disequilibrium, with persistent shortages or surpluses as the visible result.

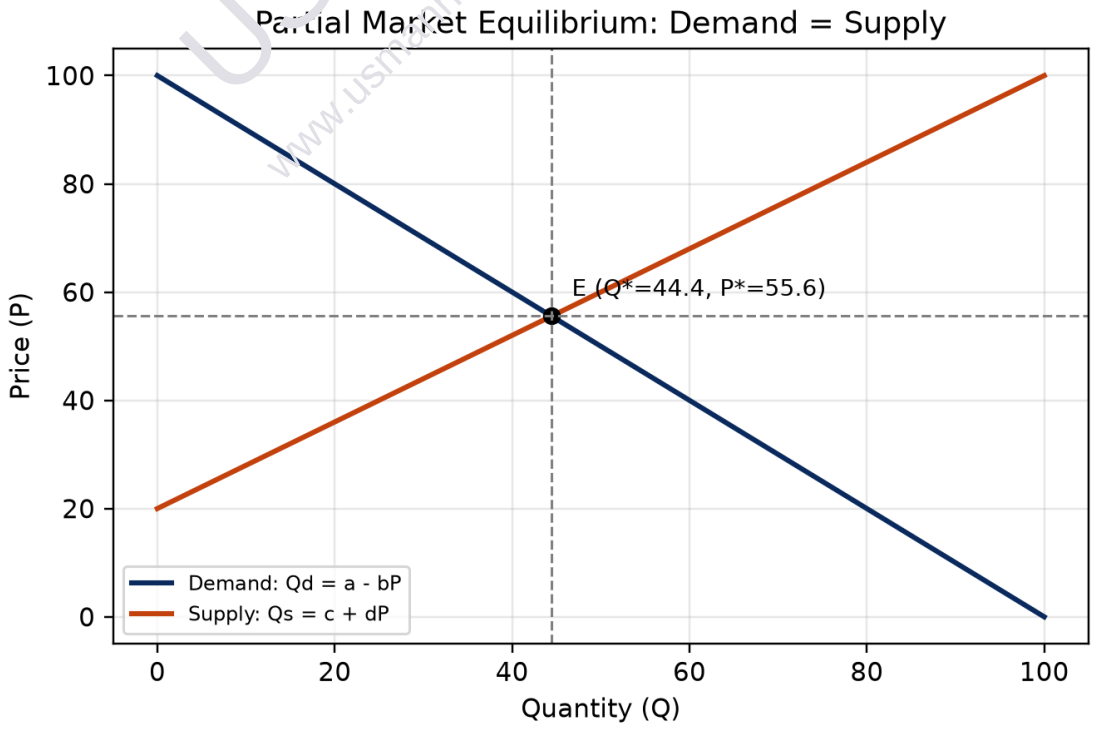


Figure 1: Partial equilibrium of a single market. The price settles where the demand curve cuts the supply curve (point E), so that quantity demanded equals quantity supplied.

Partial Market Equilibrium

Partial equilibrium analysis studies the equilibrium of a single market — or a single firm, industry or sector — in isolation, assuming that everything outside that market remains unchanged. It is the method of Marshall's Principles of Economics and of the familiar demand-and-supply diagram: the price of the good is determined by its own demand and supply, while consumers' incomes, the prices of all other goods, and conditions in factor markets are held constant under the ceteris paribus assumption. Feedback effects from the market being studied to the rest of the economy, and back again, are deliberately ignored. The method is justified when the market is small relative to the whole economy, so that a change in its price spends only a negligible fraction of consumers' incomes and causes no perceptible ripples elsewhere — the market for salt or for matches is the textbook example.

A numerical example shows the method completely. Let demand be Qd = 120 − 4P and supply be Qs = 20 + P. Partial equilibrium requires Qd = Qs, so 120 − 4P = 20 + P, giving 100 = 5P and therefore P = 20; the equilibrium quantity is Q = 20 + 20 = 40 units. If the government now fixes a maximum price of 15, quantity demanded becomes 120 − 60 = 60 while quantity supplied falls to 35, and the partial analysis immediately reveals a shortage of 25 units. The strength of the method is its simplicity and sharpness: it gives clear, testable answers about one market at a time, which is why it remains the daily tool of applied microeconomics, tax-incidence analysis and cost–benefit work.

General Market Equilibrium

General equilibrium analysis studies the equilibrium of the whole economy at once: all markets — for goods, services, labour and capital — are analysed simultaneously, and prices are determined together, each market's conditions depending on prices in every other market. The approach was founded by Léon Walras in the 1870s and later given rigorous existence proofs by Arrow and Debreu. Its starting insight is interdependence: markets are not islands. A rise in the price of petrol raises transport costs, which raises food prices, which reduces real wages, which changes the demand for labour and for petrol itself. Partial analysis assumes these echoes away; general analysis exists precisely to follow them. In a general equilibrium, a single list of prices must clear every market at the same time, and one famous result, Walras's law, states that if all markets but one are in equilibrium, the remaining market must also be in equilibrium, because the sum of the values of all excess demands is identically zero.

Because the whole system is solved together, the number of equations is enormous — in principle one demand and one supply equation for every good and factor — and the method needs matrix algebra of exactly the kind taught in this course. Its rewards are correspondingly large: it can trace the full effects of a tax, a tariff or a subsidy through every linked market, reveal unintended consequences that partial analysis would miss, and provide the framework in which the efficiency of a competitive economy (the two fundamental theorems of welfare economics) can be proved.

Table 1: Partial equilibrium versus general equilibrium compared

Point of comparison	Partial market equilibrium	General market equilibrium
Scope	One market, firm or industry	All markets of the economy together
Associated with	Alfred Marshall	Léon Walras; later Arrow and Debreu
Key assumption	Ceteris paribus – other prices and incomes constant	No such isolation; everything may adjust
Interdependence	Ignored; feedback effects neglected	Central; markets are explicitly interdependent
Number of prices determined	One price (and quantity)	The whole list of prices simultaneously
Data and mathematics needed	One demand and one supply equation; a simple diagram	Large equation systems; matrix algebra
Best suited for	Small markets; minor taxes; industry studies	Economy-wide taxes, trade policy, welfare questions
Main weakness	Misleading when the market is large or strongly linked to others	Complex, data-hungry, and hard to apply directly

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Which Method Should Be Used?

The two methods are complements, not rivals, and the competent economist chooses according to the question. If the market under study is small and the change is minor, the ignored feedbacks are second-order and partial analysis gives a quick, substantially correct answer. If the change is large or economy-wide — a general sales tax, an exchange-rate devaluation, a change in the wage level — the feedbacks are the story, and only general equilibrium reasoning can be trusted. In teaching and in this course the order is natural: master the partial equilibrium of one market first, because the general system is built by writing down exactly such market equations for every market and solving them together.

Conclusion

In conclusion, equilibrium is a balanced state in which plans are mutually consistent and no tendency to change exists, unless outside data change; it may be stable or unstable, and it need not be desirable. Partial equilibrium isolates one market under ceteris paribus and determines its price and quantity from its own demand and supply, in Marshall's tradition — simple, sharp, and valid for small markets. General equilibrium, in Walras's tradition, determines all prices simultaneously and takes the interdependence of markets as its central fact — harder, but indispensable for economy-wide questions. A sound analysis states clearly which of the two it is using, and why.

Question 3: Solving Simultaneous Equations by Cramer's Rule (20 Marks)

Using Cramer's rule, solve the following equation? 3X + 5Y = 0; 2X + 6Y = 3

Introduction

Simultaneous equations arise in economics wherever a model contains as many conditions as it has unknown variables: demand equals supply while a budget is also satisfied, two markets clear at once, or an equilibrium condition is combined with an identity. Cramer's rule is one of the most systematic ways of solving such a system. It uses determinants — single numbers computed from the coefficients of the equations — and it has two great teaching virtues: it gives each unknown by an explicit formula, so the method never gets lost in manipulation, and it shows clearly, through the value of the main determinant, whether a unique solution exists at all. This answer explains determinants and states Cramer's rule, then applies the rule step by step to the given system, and finally verifies the answer by substitution, as every numerical solution in this course should be verified.

Determinants and Cramer's Rule

For two equations in two unknowns, a1X + b1Y = c1 and a2X + b2Y = c2, the determinant of the coefficient matrix is the number D = a1b2 − a2b1. A determinant rearranges the coefficients into one value that summarises whether the equations are genuinely independent. Cramer's rule then says: form D from the coefficients of the unknowns; form DX by replacing the X-

column of coefficients with the constant terms c1 and c2; form DY by replacing the Y-column with the constant terms; the solution is then given by the ratios shown below.

X = DX / D and Y = DY / D (provided D ≠ 0)

The proviso matters. If D = 0, the rule cannot be applied because division by zero is impossible, and the system has either no solution (the equations contradict each other, like two parallel lines) or infinitely many solutions (the equations are multiples of each other, like the same line written twice). A non-zero determinant is therefore the guarantee of exactly one solution — the algebraic counterpart of two straight lines crossing at exactly one point. For larger systems the rule generalises directly: a system of n equations in n unknowns has the unique solution Xj = Dj / D, where Dj is the determinant formed by replacing the j-th column of coefficients with the constants. The rule becomes laborious for large n, which is why matrix inversion or elimination is preferred for big models, but for two or three unknowns Cramer's rule is quick, transparent and exact.

The Given System

The equations given in the question are:

3X + 5Y = 0 … (1)

2X + 6Y = 3 … (2)

Here a1 = 3, b1 = 5, c1 = 0 (from equation 1) and a2 = 2, b2 = 6, c2 = 3 (from equation 2). We now compute the three determinants, D, DX and DY, one at a time and showing every step.

Step 1: The Main Determinant D

D is built from the coefficients of X and Y exactly as they stand in the equations. Multiply the diagonal entries and subtract the product of the off-diagonal entries:

$$D = (3)(6) - (2)(5) = 18 - 10 = 8$$

Since D = 8 ≠ 0, the system possesses one and only one solution, and Cramer's rule can proceed. Had D turned out to be zero, we would have stopped here and reported the system as singular instead of inventing values.

Step 2: The Determinant DX

For DX, the column of X-coefficients (3 and 2) is removed and the constant terms (0 and 3) are put in its place; the Y-column (5 and 6) is left untouched. Notice that the zero constant of the first equation enters this determinant, which is why care with the given numbers — including their zeros — is essential:

$$DX = (0)(6) - (3)(5) = 0 - 15 = -15$$

Step 3: The Determinant DY

For DY, the Y-column is replaced by the constants while the X-column (3 and 2) stays as it was:

$$DY = (3)(3) - (2)(0) = 9 - 0 = 9$$

Step 4: The Solution

Divide each replacement determinant by the main determinant, exactly as the rule prescribes:

X = DX / D = − 15 / 8 = − 1.875

Y = DY / D = 9 / 8 = 1.125

Table 1: Summary of the three determinants and the solution

Determinant	Construction	Value
D	Coefficients of X and Y: (3)(6) − (2)(5)	8
DX	X-column replaced by constants: (0)(6) − (3)(5)	− 15
DY	Y-column replaced by constants: (3)(3) − (2)(0)	9
Solution	X = DX/D; Y = DY/D	X = − 15/8; Y = 9/8

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Verification by Substitution

A solution that is not checked is only half an answer, so we substitute X = − 15/8 and Y = 9/8 into both original equations. In equation (1): 3X + 5Y = 3(− 15/8) + 5(9/8) = − 45/8 + 45/8 = 0, which is exactly the right-hand side of equation (1). In equation (2): 2X + 6Y = 2(− 15/8) + 6(9/8) = − 30/8 + 54/8 = 24/8 = 3, which is exactly the right-hand side of equation (2). Both equations are satisfied simultaneously, so the solution is confirmed. The same result can be cross-checked by ordinary elimination for extra confidence: multiplying equation (1) by 2 and equation (2) by 3

gives 6X + 10Y = 0 and 6X + 18Y = 9; subtracting eliminates X and leaves 8Y = 9, hence Y = 9/8, and substitution then gives X = − 15/8, in complete agreement with Cramer's rule.

Why the Method Matters in Economics

Beyond this exercise, determinants and Cramer's rule recur throughout economic analysis. The equilibrium of a small general-equilibrium model, the solution of input–output (Leontief) systems, the comparative-static derivatives of an equilibrium system, and the second-order conditions for optimisation all turn on whether a certain determinant is zero or non-zero, and on its sign. Learning the rule on a two-equation system is therefore not a ritual: it is the smallest working example of the machinery with which complete economic models are solved and tested.

Conclusion

To conclude, the system 3X + 5Y = 0 and 2X + 6Y = 3 has determinant D = 8, which is non-zero, so a unique solution exists. Cramer's rule gives DX = − 15 and DY = 9, hence X = − 15/8 (≈ − 1.875) and Y = 9/8 (= 1.125). Substitution into both original equations reproduces their right-hand sides exactly, and an independent elimination check agrees, so the solution X = − 15/8, Y = 9/8 is established with full confidence.

Question 4: Derivative dy/dz by the Chain Rule (20 Marks)

Find the derivative dy/dz using the Chain rule: If $Y = 3x^2$, where $x = 2z + 3$

Introduction

Many economic relationships are chained together. Output depends on employment, and employment depends on time; demand depends on price, and price depends on a tax; consumption depends on income, and income depends on production. Whenever a variable depends on a second variable which itself depends on a third, the rate of change of the first with respect to the third cannot be read off directly — it must be built from the two links of the chain. The tool built for exactly this job is the chain rule of differentiation, one of the most frequently used rules in mathematical economics. This answer states the rule, explains why it works, applies it carefully to the given functions $Y = 3x^2$ with $x = 2z + 3$, and then confirms the result by the completely independent method of substitution, so that the two routes can be compared. A note on the interpretation: the printed question writes the final derivative as dy/dz, treating Y and y as the same dependent variable, and the inner function is read as $x = 2z + 3$ — the only reading under which the problem is a chain-rule problem at all.

The Chain Rule Stated

Suppose y is a function of x, $y = f(x)$, and x is in turn a function of z, $x = g(z)$. Then y becomes, through x, a function of z as well, and its derivative with respect to z is the product of the two derivatives along the chain:

$$\mathbf{dy/dz = (dy/dx) \times (dx/dz)}$$

The intuition is that rates of change multiply. If y changes three times as fast as x, and x changes twice as fast as z, then y plainly changes six times as fast as z. In Leibniz's notation the rule looks almost like cancelling dx from a fraction, which makes it easy to remember, although

derivatives are not literally fractions and the rule is properly proved as a limit statement. The rule extends to chains of any length: for y depending on x, x on w, and w on z, the derivative dy/dz is the product of the three derivatives dy/dx, dx/dw and dw/dz. In economic language, the chain rule is the mathematics of indirect effects: a change in z reaches y only through the intermediate variable, and the total response is the direct sensitivity of y to x multiplied by the sensitivity of x to z.

Applying the Rule to the Given Functions

The two links of the chain in this question are $Y = 3x^2$, the outer function, and $x = 2z + 3$, the inner function. We differentiate each link separately first. For the outer function, the power rule — bring the power down and reduce it by one — gives:

$$\mathbf{dY/dx = 3 \times 2x = 6x}$$

For the inner function, x is a linear function of z; the derivative of 2z is 2 and the derivative of the constant 3 is zero, so:

$$\mathbf{dx/dz = 2}$$

Now multiply the two links together, as the chain rule instructs:

$$\mathbf{dY/dz = (dY/dx) \times (dx/dz) = 6x \times 2 = 12x}$$

This answer still contains the intermediate variable x, which is correct chain-rule practice at this stage but not yet the final form, because the question asks for the rate of change with respect to z. The finishing step is therefore to express the result wholly in terms of z by substituting the inner function x = 2z + 3 into it:

$$dY/dz = 12x = 12(2z + 3) = 24z + 36$$

Thus the derivative is dY/dz = 24z + 36. Notice the structure of this result: it is not a constant, because the outer function $Y = 3x^2$ is a curve, not a straight line — its slope 6x grows as x grows, and since x itself grows with z, the overall rate of change grows with z as well. At z = 0, for instance, x = 3 and the rate of change is 36; at z = 1, x = 5 and the rate of change is 60, which agrees with 6x × 2 = 30 × 2 at x = 5. Reading the result at particular points in this way is a useful habit, because it converts an abstract formula into concrete, checkable numbers.

Check by Direct Substitution

The chain rule's answer can be verified — and the rule itself made more convincing — by removing the chain before differentiating. Substitute $x = 2z + 3$ into $Y = 3x^2$ at the very start. First square the bracket: $(2z + 3)^2 = 4z^2 + 12z + 9$. Multiply through by 3:

$$Y = 3(2z + 3)^2 = 12z^2 + 36z + 27$$

Now Y is an ordinary function of z alone, and the elementary power rule applies term by term. The derivative of $12z^2$ is 24z, the derivative of 36z is 36, and the derivative of the constant 27 is zero:

$$dY/dz = 24z + 36$$

The two methods agree exactly, as they must. The agreement is worth dwelling on, because it shows what the chain rule really is: a shortcut that produces, without the labour of expanding brackets, precisely the result that substitution and ordinary differentiation would give. For a single power the expansion is easy, but for a function like $Y = (2z + 3)^{10}$ or Y = e raised to a

complicated power, expanding first would be painful or impossible, while the chain rule remains a two-line calculation. That is why economists reach for it constantly.

Common Mistakes and How to Avoid Them

Three errors recur in chain-rule questions and are worth guarding against. The first is to differentiate only the outer function and forget the inner derivative altogether — writing dY/dz = 6x and stopping, which omits the factor dx/dz = 2 and understates the answer by half. The second is to differentiate the outer function as though its variable were z, mixing the two variables inconsistently; discipline in writing dY/dx and dx/dz separately, as this answer has done, prevents it. The third is to leave the final answer in terms of the intermediate variable when the question asks for the derivative with respect to the original variable; the substitution step that converts 12x into 24z + 36 is part of the answer, not an optional decoration. A useful final check on any chain-rule result is the substitution method used above: if time permits, perform it, and require the two results to match.

Economic Illustration

To fix the idea in economic terms, suppose z is time in years, x is a firm's capital stock growing according to x = 2z + 3 (in suitable units), and output is Y = $3x^2$. The chain rule then says that output grows over time at the rate 24z + 36 units per year — slowly at first and faster later, because both links of the chain reinforce each other: the capital stock is rising, and each extra unit of capital is worth more output than the last under this production function. Indirect chains of exactly this kind — a variable moving because a variable behind it is moving — are the rule

rather than the exception in economics, from inflation passing through costs into prices to exchange rates passing through import prices into the cost of living.

Conclusion

In conclusion, with $Y = 3x^2$ and $x = 2z + 3$, the chain rule gives $dY/dz = (dY/dx)(dx/dz) = 6x \times 2 = 12x$, and substituting the inner function converts this into the final answer $dY/dz = 24z + 36$. Direct substitution before differentiating produces $Y = 12z^2 + 36z + 27$ and the identical derivative, confirming the result. The example displays the whole logic of the chain rule: differentiate each link, multiply the links, and express the answer in the variable originally asked about.

Question 5: The Total Derivative dy/dt (20 Marks)

Find the total derivative dy/dt: $Y = 2x^2 - t^3$, where $x = f(t) = 2t^3 + 3t$

Introduction

In the previous questions, derivatives measured how one variable responds when one other variable changes. Real economic variables, however, usually move for more than one reason at the same time. In the function given here, $Y = 2x^2 - t^3$, the variable t reaches Y along two distinct routes: directly, through the term $-t^3$ that contains t itself, and indirectly, because x is a function of t, $x = 2t^3 + 3t$, and Y depends on x. The total derivative is the tool that adds both routes together and so measures the complete change in Y when t changes. It is one of the most important ideas of the course, because comparative statics — asking how an equilibrium value changes when a parameter changes — is nothing but total differentiation applied to an

economic model. This answer explains the idea, states the total-derivative formula, applies it step by step to the given function, and confirms the answer by direct substitution.

Partial Effects and the Total Effect

Hold x fixed for a moment and ask how Y responds to t alone. the term $2x^2$ does not move, and the term $-t^3$ contributes $-3t^2$. This is the direct, or partial, effect of t on Y, written $\partial Y/\partial t = -3t^2$. Now hold the direct channel aside and ask how Y responds when x moves: dY/dx for $Y = 2x^2$ is $4x$. But x does not move freely — it is tied to t by $x = 2t^3 + 3t$, whose derivative is $dx/dt = 6t^2 + 3$. So when t changes, x changes at the rate $6t^2 + 3$, and this movement of x pushes Y at the rate $4x$ per unit of x. The indirect effect of t on Y through x is therefore the product $4x \times (6t^2 + 3)$, which the reader will recognise as a chain-rule term, exactly as in Question 4. The total derivative is the sum of the direct effect and the indirect effect:

$$\mathbf{dY/dt = (\partial Y/\partial x)(dx/dt) + (\partial Y/\partial t)}$$

The notation deserves a sentence. A straight d is used on the left, dY/dt, because in the end Y is wholly a function of the single variable t and its rate of change is an ordinary complete derivative. Curved ∂ symbols are used for the pieces, because each piece holds one channel in isolation while the other is mentally frozen. The formula generalises naturally: if Y depends on several variables which each depend on t, there is one chain term for every channel, plus the direct term if t also appears in Y itself. The total differential $dY = (\partial Y/\partial x)dx + (\partial Y/\partial t)dt$ is the same statement written for small changes rather than rates, and dividing it through by dt gives the formula above.

Step-by-Step Working

We now evaluate each piece of the formula for the given functions. The pieces are:

- Partial derivative with respect to x: from $Y = 2x^2 - t^3$, treating t as a constant, $\partial Y/\partial x = 4x$. (The term $-t^3$ vanishes because, with t held fixed, it is a constant.)
- Derivative of the inner function: from $x = 2t^3 + 3t$, $dx/dt = 6t^2 + 3$.
- Partial derivative with respect to t: from $Y = 2x^2 - t^3$, treating x as a constant, $\partial Y/\partial t = -3t^2$. (The term $2x^2$ vanishes because, with x held fixed, it is a constant.)

Assemble the pieces in the formula:

$$dY/dt = 4x(6t^2 + 3) + (-3t^2) = 4x(6t^2 + 3) - 3t^2$$

The answer is not finished until the intermediate variable x has been removed, because the question asks for a derivative with respect to t and the result should therefore be a function of t alone. Substitute $x = 2t^3 + 3t$:

$$dY/dt = 4(2t^3 + 3t)(6t^2 + 3) - 3t^2$$

Now expand the product carefully. First, $4(2t^3 + 3t) = 8t^3 + 12t$. Multiply by $(6t^2 + 3)$: $(8t^3)(6t^2) = 48t^5$; $(8t^3)(3) = 24t^3$; $(12t)(6t^2) = 72t^3$; $(12t)(3) = 36t$. Collecting, $24t^3 + 72t^3 = 96t^3$, so the indirect effect alone is $48t^5 + 96t^3 + 36t$. Adding the direct effect $-3t^2$ gives the total derivative in its final form:

$$dY/dt = 48t^5 + 96t^3 - 3t^2 + 36t$$

Check by Direct Substitution

As with the chain rule, an independent check is available: substitute $x = 2t^3 + 3t$ into Y before differentiating, so that Y becomes a function of t alone, and then use ordinary rules. Squaring first, $(2t^3 + 3t)^2 = 4t^6 + 12t^4 + 9t^2$, and doubling gives $2x^2 = 8t^6 + 24t^4 + 18t^2$. Hence:

$$Y = 8t^6 + 24t^4 + 18t^2 - t^3$$

Differentiate term by term: $8t^6$ gives $48t^5$; $24t^4$ gives $96t^3$; $18t^2$ gives $36t$; and $-t^3$ gives $-3t^2$. Collecting the four terms in any order reproduces exactly $48t^5 + 96t^3 - 3t^2 + 36t$. The agreement between the total-derivative formula and brute-force substitution confirms both the method and the arithmetic. A quick numerical spot check underlines the point: at $t = 1$, the formula gives $dY/dt = 48 + 96 - 3 + 36 = 177$; the pieces give indirect effect $4x(6t^2 + 3)$ with $x = 5$, that is $20 \times 9 = 180$, plus direct effect -3, totalling 177 — the same number by both routes.

Why the Total Derivative Matters

The distinction between a partial effect and a total effect is not pedantry; confusing them produces genuinely wrong economics. Suppose a tax rate t rises and revenue depends both on the rate directly and on the tax base, which itself shrinks as the rate rises. The partial effect — more revenue per unit of base — is positive, but the total effect includes the negative indirect effect through the shrinking base, and may be smaller, zero, or even negative: this is the logic behind the Laffer curve, and it is exactly the structure of the present question, with one positive indirect term and one negative direct term combined in a single derivative. Likewise, in equilibrium models, differentiating an equilibrium condition totally with respect to a parameter is the standard method of comparative statics — the method by which economists derive how

price changes when demand shifts, or how income changes when government spending rises, allowing every variable in the model to adjust as it does in reality.

One caution should be recorded. The total-derivative formula assumes that the intermediate function x = f(t) is known and given. If instead x were determined inside a model by an equilibrium condition, the same formula still applies, but dx/dt would itself have to be found by differentiating that condition — which is precisely the comparative-statics procedure just described. The present question, in which the function f is supplied in full, is the clean training case for that more advanced use.

Conclusion

In conclusion, Y depends on t through two channels — directly through $-t^3$ and indirectly through $x = 2t^3 + 3t$ — and the total derivative adds them: $dY/dt = (\partial Y/\partial x)(dx/dt) + \partial Y/\partial t = 4x(6t^2 + 3) - 3t^2$. Substituting for x and expanding gives the final answer $dY/dt = 48t^5 + 96t^3 - 3t^2 + 36t$, which direct substitution and a numerical spot check both confirm exactly.