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Name:

Registration No.:

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Program: BS

Semester: Autumn 2026

ALLAMA IQBAL OPEN UNIVERSITY, ISLAMABAD

(Department of Computer Science)

Course: Pre-Calculus I (3517) • Level: BS

Semester: Autumn 2026

Assignment No. 2 (Units 5-9)

Total Marks: 100 • Pass Marks: 50

SOLVED ASSIGNMENT GUIDE — For Study & Reference

Note: This guide is prepared for study and reference purposes only. Students are advised to read the relevant units of the course book, understand every definition, theorem and step of working, practise each type of question by hand, and then write the assignment in their own words and in their own handwriting or typing. Copying this guide word for word is not allowed. Every calculation in this guide has been checked by substitution or by an independent method, and students should repeat those checks themselves before submitting.

Question 1: Solving Linear Systems by Matrix Inversion and by Gaussian Elimination (10+10)

Question as printed: (a) Use matrix inversion to solve the system: 2x + 3y = 13, 5x − y = 4. Verify your solution. (b) Solve using Gaussian elimination: x + y − z = 2, 2x − y + 3z = 9, x + 2y + z = 8.

Introduction

Systems of linear equations are the first major application of matrices in this course, and this question presents the two standard systematic methods side by side. Matrix inversion rewrites the system as AX = B and solves it in one stroke as $X = A^{-1}B$; it is elegant, it displays the condition for a unique solution (the determinant must be non-zero), and it is ideal when the same coefficient matrix must be used with many different right-hand sides. Gaussian elimination is the more general workhorse: it reduces the augmented matrix by elementary row operations to upper triangular (row echelon) form and then recovers the unknowns by back substitution; it works for any number of equations and unknowns and also detects inconsistent and under-determined systems. The two systems here are deliberately different in size — two equations in two unknowns for inversion, three in three for elimination — and, as the answers show, both have fractional solutions, which is normal in real problems and is exactly why the verification step demanded in part (a), and repeated by good practice in part (b), matters: substitution back into the original equations is the proof that no arithmetic slip has occurred.

Part (a): Solution by Matrix Inversion

Write the system 2x + 3y = 13, 5x − y = 4 in matrix form AX = B with A = [[2, 3], [5, −1]], $X = [x, y]^T$ and $B = [13, 4]^T$. Step 1 — Determinant: det A = (2)(−1) − (3)(5) = −2 − 15 = −17. Since det A = −17 ≠ 0, the inverse exists and the system has exactly one solution. Step 2 — Inverse: for a 2×2 matrix [[a, b], [c, d]], the inverse is (1/(ad − bc)) [[d, −b], [−c, a]]. Applying this, A^{-1} = (1/−17) [[−1, −3], [−5, 2]] = (1/17) [[1, 3], [5, −2]]. As a safety check, $A^{-1}A$ should be the identity: (1/17) [[1, 3], [5, −2]] [[2, 3], [5, −1]] = (1/17) [[2 + 15, 3 − 3], [10 − 10, 15 + 2]] = (1/17) [[17, 0], [0, 17]]

= I, as required. Step 3 — Solve: $X = A^{-1}B = (1/17)\ [[1, 3], [5, -2]]\ [13, 4]^T = (1/17)\ [1(13) + 3(4), 5(13) - 2(4)]^T = (1/17)\ [25, 57]^T$. Hence x = 25/17 and y = 57/17 (approximately x ≈ 1.47, y ≈ 3.35). Step 4 — Verification, as the question requires: substituting into the first equation, 2(25/17) + 3(57/17) = (50 + 171)/17 = 221/17 = 13 ✓, substituting into the second equation, 5(25/17) − 57/17 = (125 − 57)/17 = 68/17 = 4 ✓. Both original equations are satisfied exactly, so the solution is confirmed. Note that the fractions are genuine features of this system, not errors: the determinant −17 in the denominator of A^{-1} propagates into the answer, which is typical whenever the determinant is not ±1.

Table 1: Matrix inversion at a glance

Quantity	Value
Coefficient matrix A	[[2, 3], [5, −1]]
det A	−17 (non-zero, so a unique solution exists)
A inverse	(1/17) [[1, 3], [5, −2]]
Solution	x = 25/17, y = 57/17
Check in 2x + 3y = 13	221/17 = 13 ✓
Check in 5x − y = 4	68/17 = 4 ✓

Part (b): Solution by Gaussian Elimination

Form the augmented matrix of the system x + y − z = 2; 2x − y + 3z = 9; x + 2y + z = 8. The starting matrix has rows R1 = [1, 1, −1 | 2], R2 = [2, −1, 3 | 9], R3 = [1, 2, 1 | 8]. The aim of the forward phase is to create zeros below the leading 1 in the first column. Operation 1: R2 → R2 − 2R1 gives R2 = [0, −3, 5 | 5]. Operation 2: R3 → R3 − R1 gives R3 = [0, 1, 2 | 6]. The matrix is

now [1, 1, −1 | 2; 0, −3, 5 | 5; 0, 1, 2 | 6]. For convenience interchange R2 and R3 so that the middle pivot is 1: rows become R2 = [0, 1, 2 | 6], R3 = [0, −3, 5 | 5]. Operation 3: R3 → R3 + 3R2 gives R3 = [0, 0, 11 | 23], completing the triangular form: [1, 1, −1 | 2; 0, 1, 2 | 6; 0, 0, 11 | 23]. Back substitution now proceeds from the last row upwards. From R3: 11z = 23, so z = 23/11. From R2: y + 2z = 6, so y = 6 − 2(23/11) = 6 − 46/11 = (66 − 46)/11 = 20/11. From R1: x + y − z = 2, so x = 2 − y + z = 2 − 20/11 + 23/11 = 2 + 3/11 = 25/11. The solution is x = 25/11, y = 20/11, z = 23/11 (approximately 2.27, 1.82, 2.09). Verification in all three original equations: (i) 25/11 + 20/11 − 23/11 = 22/11 = 2 ✓; (ii) 2(25/11) − 20/11 + 3(23/11) = (50 − 20 + 69)/11 = 99/11 = 9 ✓; (iii) 25/11 + 2(20/11) + 23/11 = (25 + 40 + 23)/11 = 88/11 = 8 ✓. All three equations check exactly, so the system has the unique solution (25/11, 20/11, 23/11). The elimination involved no row scaling by zero and no contradictory row such as [0, 0, 0 | nonzero], confirming consistency and uniqueness — the coefficient determinant is in fact −11, which is why elevenths appear, just as seventeenths appeared in part (a).

Table 2: Gaussian elimination — the augmented matrix at each stage

Stage	Row 1	Row 2	Row 3
Start	1, 1, −1 | 2	2, −1, 3 | 9	1, 2, 1 | 8
After R2 − 2R1 and R3 − R1	1, 1, −1 | 2	0, −3, 5 | 5	0, 1, 2 | 6
After swapping R2, R3 and R3 + 3R2	1, 1, −1 | 2	0, 1, 2 | 6	0, 0, 11 | 23
Back substitution result	x = 25/11	y = 20/11	z = 23/11

Conclusion

Matrix inversion gives x = 25/17, y = 57/17 for the 2×2 system, and Gaussian elimination gives x = 25/11, y = 20/11, z = 23/11 for the 3×3 system; both solutions are verified by exact substitution into every original equation. The comparison teaches the practical rule of the unit: inversion is clean and instructive for small systems and repeated right-hand sides, while elimination is the method that scales, needs no determinant computation in advance, and reveals inconsistent or dependent systems through the form of its rows. In both methods the fractional answers are correct and expected, and the verification step — not the method — is what guarantees them.

Question 2: Arithmetic Means and an Infinite Geometric Series (10+10)

Question as printed: (a) Insert four arithmetic means between 3 and 23. Also find the sum of all six terms. (b) The sum of an infinite geometric series is 12, and the first term is 4. Find the common ratio and write the first five terms of the series.

Introduction

Sequences and series organise numbers that follow a rule of growth, and the two most important rules are the two in this question. An arithmetic progression (AP) grows by adding the same difference at each step; a geometric progression (GP) grows by multiplying by the same ratio at each step. Part (a) asks for the four arithmetic means between two given numbers — the four numbers that must be inserted so that the whole list of six is an AP — and

for the sum of the resulting six terms, which the AP sum formula gives in one line once the progression is known. Part (b) reverses the usual direction of a GP problem: instead of being given the ratio and asked for the sum to infinity, we are given the sum to infinity and the first term and must recover the ratio, using the formula $S = a/(1 - r)$, which is valid exactly when $|r| < 1$. The answer then has to be checked to satisfy that condition — a step students often omit, even though a 'sum to infinity' for a series with $|r| \geq 1$ would be meaningless because the series would diverge.

Part (a): Four Arithmetic Means Between 3 and 23

Inserting four arithmetic means between 3 and 23 means finding four numbers A1, A2, A3, A4 such that 3, A1, A2, A3, A4, 23 is an arithmetic progression. The progression therefore has six terms in total: first term $a = 3$, last term (sixth term) 23, number of terms $n = 6$. If d is the common difference, the nth-term formula $a_n = a + (n - 1)d$ with $n = 6$ gives $23 = 3 + 5d$, so $5d = 20$ and $d = 4$. The inserted means are obtained by repeatedly adding 4: A1 = 3 + 4 = 7, A2 = 7 + 4 = 11, A3 = 11 + 4 = 15, A4 = 15 + 4 = 19, and indeed the next step gives 19 + 4 = 23, the given last term, which checks the work. The full progression is 3, 7, 11, 15, 19, 23. For the sum of all six terms, use the AP sum formula $S_n = (n/2)(\text{first term} + \text{last term})$: $S_6 = (6/2)(3 + 23) = 3 \times 26 = 78$. Direct addition confirms it: 3 + 7 + 11 + 15 + 19 + 23 = 78. Two useful observations follow. The general recipe for inserting m means between two numbers p and q is $d = (q - p)/(m + 1)$ — here $(23 - 3)/(4 + 1) = 4$ — because the inserted means create $m + 1$ equal steps between the endpoints. And the average of the six terms, 78/6 = 13, equals the average of the first and last terms, (3 + 23)/2 = 13, an instance of the general AP fact that the mean of the whole

progression equals the mean of its extremes; it also equals the middle average (11 + 15)/2 = 13, illustrating the symmetry of an AP about its centre.

Table 1: The arithmetic progression with four means inserted between 3 and 23

Position	1st (given)	Mean 1	Mean 2	Mean 3	Mean 4	6th (given)
Term	3	7	11	15	19	23
Running total	3	10	21	36	55	78

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Part (b): Recovering the Ratio from the Sum to Infinity

For an infinite geometric series with first term a and common ratio r, the sum to infinity is S = a/(1 − r), valid if and only if |r| < 1. Here a = 4 and S = 12. Substituting, 12 = 4/(1 − r), so 1 − r = 4/12 = 1/3, and therefore r = 1 − 1/3 = 2/3. The required condition is satisfied, since |2/3| < 1, so the series genuinely converges to 12 and the data are consistent. (Had the computation produced |r| ≥ 1, the correct conclusion would have been that no such convergent series exists.) The terms are generated by $a_n = a \cdot r^{n-1} = 4(2/3)^{n-1}$. The first five terms are: a1 = 4; a2 = 4(2/3) = 8/3 ≈ 2.667; a3 = $4(2/3)^2$ = 16/9 ≈ 1.778; a4 = $4(2/3)^3$ = 32/27 ≈ 1.185; a5 = $4(2/3)^4$ = 64/81 ≈ 0.790. As a check on convergence, the partial sums climb towards 12 without ever exceeding it: S1 = 4, S2 = 20/3 ≈ 6.67, S3 = 76/9 ≈ 8.44, S4 = 260/27 ≈ 9.63, S5 = 844/81 ≈ 10.42, and the remaining tail after five terms is S − S5 = $a \cdot r^5/(1 - r)$ = 4(32/243)/(1/3) = 128/81 ≈ 1.58, which added to 10.42 gives 12.00 as expected (exactly, 844/81 + 128/81 = 972/81 = 12). The terms shrink towards zero — the necessary condition for convergence — and each term is exactly two-thirds of its predecessor, as the definition of a GP with ratio 2/3 demands.

Table 2: First five terms of the geometric series and the running sums

n	Term $a_n = 4(2/3)^{n-1}$	Decimal	Partial sum S_n
1	4	4.000	4.000
2	8/3	2.667	6.667
3	16/9	1.778	8.444
4	32/27	1.185	9.630
5	64/81	0.790	10.420
∞	→ 0	—	12 (given sum)

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Common Errors to Avoid in This Question

Three mistakes account for most lost marks on questions of this type, and both parts of this answer have been written to avoid them. The first is miscounting the terms when means are inserted: four means between two endpoints make six terms and five equal steps, not four steps, so the divisor in d = (q − p)/(m + 1) is 5 here, not 4; using 4 would give the wrong difference 5 and a progression ending at 28 instead of 23, which the final-step check (19 + 4 = 23) immediately exposes. The second is applying the sum-to-infinity formula mechanically without checking |r| < 1: the formula is a theorem about convergent series only, and part (b) therefore verifies that r = 2/3 satisfies the condition before the terms are written down. The third is rounding the fractional geometric terms too early; keeping 8/3, 16/9, 32/27 and 64/81 as exact fractions makes the partial sums and the tail calculation come out to exactly 12, whereas rounded decimals accumulate visible error. Writing the general formula first, substituting second, and checking last is the order that prevents all three errors.

Conclusion

Part (a): with common difference 4, the four arithmetic means between 3 and 23 are 7, 11, 15 and 19, and the six terms 3, 7, 11, 15, 19, 23 sum to 78. Part (b): the formula S = a/(1 − r) with a = 4 and S = 12 gives the common ratio r = 2/3, which satisfies $|r| < 1$ as it must, and the first five terms are 4, 8/3, 16/9, 32/27 and 64/81, with partial sums approaching 12 exactly as convergence requires. The two parts display the characteristic contrast of the unit: arithmetic growth is linear and its finite sums pair up symmetrically, while geometric growth with a ratio below 1 in absolute value can accumulate infinitely many terms into a finite total.

Question 3: Angles of a Triangle by the Law of Cosines, and the Fundamental Trigonometric Identity (10+10)

Question as printed: (a) In triangle ABC, a = 7, b = 9, c = 11. Use the law of cosines to find all three angles. (b) Prove the identity: $\sin^2\theta + \cos^2\theta = 1$. Hence derive expressions for $\sec^2\theta$ and $\operatorname{cosec}^2\theta$.

Introduction

Part (a) is the SSS problem of triangle solving: all three sides are given and all three angles are required. The law of cosines is the tool designed for exactly this case, because each version of it expresses the cosine of one angle purely in terms of the three sides. Two checks govern the work: the triangle inequality must hold for a genuine triangle to exist (here 7 + 9 = 16 > 11, 7 + 11 > 9 and 9 + 11 > 7, so it does), and the three computed angles must sum to 180°, which catches arithmetic errors. The convention throughout is the standard one: side a is opposite

angle A, side b opposite B, and side c opposite C, so the largest side, c = 11, must face the largest angle, C — a useful prediction to test at the end. Part (b) asks for the proof of the most fundamental identity in trigonometry, $\sin^2\theta + \cos^2\theta = 1$, which is nothing but Pythagoras' theorem in disguise, and then for the two identities obtained from it by division, which give $\sec^2\theta$ and $\operatorname{cosec}^2\theta$ in terms of $\tan^2\theta$ and $\cot^2\theta$. These three identities are used constantly in simplifying trigonometric expressions and later in calculus.

Part (a): Finding Angles A, B and C from a = 7, b = 9, c = 11

The law of cosines states $c^2 = a^2 + b^2 - 2ab\cos C$, with cyclic versions for the other angles; solving for the cosine gives $\cos A = (b^2 + c^2 - a^2)/(2bc)$, $\cos B = (a^2 + c^2 - b^2)/(2ac)$ and $\cos C = (a^2 + b^2 - c^2)/(2ab)$. Angle A: $\cos A = (9^2 + 11^2 - 7^2)/(2\cdot9\cdot11) = (81 + 121 - 49)/198 = 153/198 \approx 0.7727$, so $A = \cos^{-1}(0.7727) \approx 39.4°$. Angle B: $\cos B = (7^2 + 11^2 - 9^2)/(2\cdot7\cdot11) = (49 + 121 - 81)/154 = 89/154 \approx 0.5779$, so $B = \cos^{-1}(0.5779) \approx 54.7°$. Angle C: $\cos C = (7^2 + 9^2 - 11^2)/(2\cdot7\cdot9) = (49 + 81 - 121)/126 = 9/126 \approx 0.0714$, so $C = \cos^{-1}(0.0714) \approx 85.9°$. Now apply the checks. Sum: $39.4° + 54.7° + 85.9° = 180.0°$, exactly as a triangle requires (small rounding at the first decimal is expected; carrying more digits, A ≈ 39.39°, B ≈ 54.72°, C ≈ 85.89°, summing to 180.00°). Ordering: the angles increase as A < B < C, matching the side ordering a < b < c, and the largest angle C ≈ 85.9°, opposite the longest side, is just short of a right angle — consistent with cos C being small and positive, and with $c^2 = 121$ being slightly less than $a^2 + b^2 = 130$, which by the converse of Pythagoras makes the triangle acute-angled at C; indeed all three cosines are positive, so the triangle is acute throughout. In practice, once two angles have been found from the law of cosines, the third can equally well be obtained from the angle sum, and the agreement of the two routes is itself a verification; a cautious solver also prefers the law of

cosines over the law of sines for the largest angle, because $\cos^{-1}$ distinguishes acute from obtuse unambiguously, whereas $\sin^{-1}$ does not.

Table 1: Law-of-cosines computation for a = 7, b = 9, c = 11

Angle	Cosine formula value	Angle (degrees)
A (opposite a = 7)	(81 + 121 − 49)/198 = 153/198 ≈ 0.7727	≈ 39.4°
B (opposite b = 9)	(49 + 121 − 81)/154 = 89/154 ≈ 0.5779	≈ 54.7°
C (opposite c = 11)	(49 + 81 − 121)/126 = 9/126 ≈ 0.0714	≈ 85.9°
Sum	—	180.0° ✓

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Triangle ABC with a = 7, b = 9, c = 11 (drawn to scale)

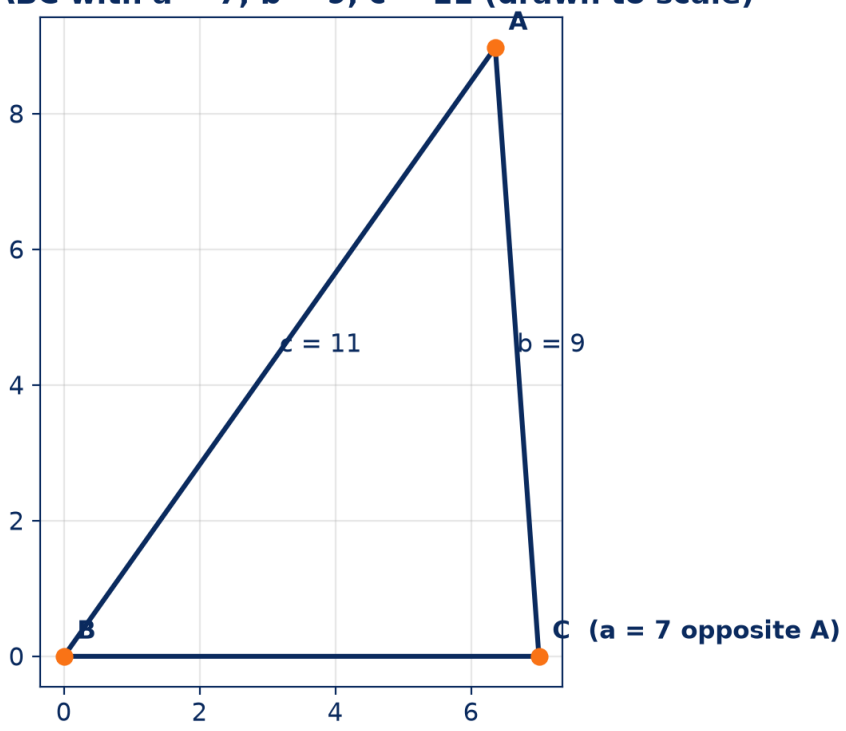


Figure 1: Triangle ABC with sides a = 7, b = 9, c = 11 drawn to scale.

Part (b): Proof of $\sin^2\theta + \cos^2\theta = 1$ and the Derived Identities

Proof: Let θ be an acute angle in a right-angled triangle with hypotenuse h, side opposite θ of length p (perpendicular) and side adjacent to θ of length b (base). By the definitions of sine and cosine, $\sin\theta = p/h$ and $\cos\theta = b/h$. Then $\sin^2\theta + \cos^2\theta = p^2/h^2 + b^2/h^2 = (p^2 + b^2)/h^2$. By Pythagoras' theorem applied to the same triangle, $p^2 + b^2 = h^2$, so $\sin^2\theta + \cos^2\theta = h^2/h^2 = 1$. This proves the identity for acute angles. For a general angle, the unit-circle definition gives cos θ and sin θ as the x- and y-coordinates of the point where the terminal side of θ meets the circle $x^2 + y^2 = 1$; the point lies on the circle, so its coordinates satisfy $x^2 + y^2 = 1$, which is precisely $\cos^2\theta + \sin^2\theta = 1$ — so the identity holds for every angle θ, acute, obtuse, reflex or negative. Now derive the companion identities by dividing the proved identity by $\cos^2\theta$ and by $\sin^2\theta$ respectively (legitimate wherever the divisor is non-zero). Dividing $\sin^2\theta + \cos^2\theta = 1$ throughout by $\cos^2\theta$ (θ not an odd multiple of 90°, so that $\cos\theta \neq 0$) gives $\tan^2\theta + 1 = \sec^2\theta$, that is, $\sec^2\theta = 1 + \tan^2\theta$, using $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$. Dividing instead throughout by $\sin^2\theta$ (θ not a multiple of 180°, so that $\sin\theta \neq 0$) gives $1 + \cot^2\theta = \operatorname{cosec}^2\theta$, that is, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$, using $\cos\theta/\sin\theta = \cot\theta$ and $1/\sin\theta = \operatorname{cosec}\theta$. These are the required expressions for $\sec^2\theta$ and $\operatorname{cosec}^2\theta$. Equivalently, directly from the reciprocals, $\sec^2\theta = 1/\cos^2\theta$ and $\operatorname{cosec}^2\theta = 1/\sin^2\theta$. Numerical checks confirm all three identities: at θ = 45°, $\sin^2\theta + \cos^2\theta = 1/2 + 1/2 = 1$ and $\sec^2\theta = (\sqrt{2})^2 = 2 = 1 + \tan^2 45° = 1 + 1$ ✓; at θ = 30°, $\operatorname{cosec}^2\theta = (2)^2 = 4 = 1 + \cot^2 30° = 1 + (\sqrt{3})^2 = 4$ ✓; while $\sin^2 30° + \cos^2 30° = 1/4 + 3/4 = 1$ ✓.

Table 2: The three Pythagorean identities

Identity	Obtained by	Valid where
$\sin^2\theta + \cos^2\theta = 1$	Pythagoras / unit circle	All θ
$\sec^2\theta = 1 + \tan^2\theta$	Dividing by $\cos^2\theta$	$\cos\theta \neq 0$
$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$	Dividing by $\sin^2\theta$	$\sin\theta \neq 0$

Conclusion

In part (a), the law of cosines gives A ≈ 39.4°, B ≈ 54.7° and C ≈ 85.9°, summing to 180° and ordered consistently with the sides, so the triangle is completely solved. In part (b), $\sin^2\theta + \cos^2\theta = 1$ is proved from Pythagoras' theorem (and extended to all angles by the unit circle), and division by $\cos^2\theta$ and $\sin^2\theta$ yields $\sec^2\theta = 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ respectively, each verified numerically. Together the parts show trigonometry's two working modes: computing with triangles, and proving relations that hold identically for all angles.

Question 4: The Hyperbola $16x^2 - 9y^2 = 144$ and a Circle Through Three Points (10+10)

Question as printed: (a) For the hyperbola $16x^2 - 9y^2 = 144$, find the center, vertices, foci, eccentricity, and equations of asymptotes. Sketch it. (b) Find the equation of the circle passing through (1, 0), (0, 1), and (2, 2). State its center and radius.

Introduction

Conic sections — circle, parabola, ellipse and hyperbola — are the curves obtained by slicing a cone, and algebraically they are exactly the curves whose equations are of the second degree in

x and y. This question takes one hyperbola and one circle and asks for their complete identifying data. For the hyperbola the method is always the same: divide through to force the right-hand side to be 1, read off a^2 and b^2 from the denominators, compute c from $c^2 = a^2 + b^2$ (the hyperbola relation, to be contrasted with the ellipse relation $c^2 = a^2 - b^2$), and then state centre, vertices, foci, eccentricity and asymptotes. For the circle through three points the method is to use the general second-degree equation with no xy term and equal coefficients of x^2 and y^2, substitute the three points to obtain three linear equations in the unknown coefficients, and solve — or equivalently, to find the centre as the intersection of the perpendicular bisectors of two chords. In both parts the results will be verified, and the sketches described, since a conic answer without its key features checked is only half an answer.

Part (a): Analysis of the Hyperbola $16x^2 - 9y^2 = 144$

Divide both sides by 144 to reach standard form: $x^2/9 - y^2/16 = 1$, that is, $x^2/3^2 - y^2/4^2 = 1$. Comparing with $x^2/a^2 - y^2/b^2 = 1$ gives $a^2 = 9$ so $a = 3$, and $b^2 = 16$ so $b = 4$. Because the x^2 term is positive, the transverse axis is the x-axis and the hyperbola opens left and right. Centre: there are no linear terms in x or y, so the centre is the origin (0, 0). Focal distance: $c^2 = a^2 + b^2 = 9 + 16 = 25$, so $c = 5$. Vertices: the points $(\pm a, 0) = (3, 0)$ and $(-3, 0)$, where the two branches come closest to the centre; the transverse axis has length $2a = 6$. Co-vertices: $(0, \pm b) = (0, 4)$ and $(0, -4)$, the endpoints of the conjugate axis of length $2b = 8$ (these points are not on the curve; they mark the 'central rectangle' of sides 2a by 2b through whose corners the asymptotes pass). Foci: $(\pm c, 0) = (5, 0)$ and $(-5, 0)$. Eccentricity: $e = c/a = 5/3 \approx 1.67$, which is greater than 1 as it must be for a hyperbola — the larger e, the more 'open' the branches. Asymptotes: the pair of

lines $y = \pm(b/a)x$, that is, $y = (4/3)x$ and $y = -(4/3)x$, equivalently $4x - 3y = 0$ and $4x + 3y = 0$; each branch approaches one pair of these lines ever more closely without ever touching them. For completeness, the directrices are $x = \pm a/e = \pm 9/5$, and the latus rectum has length $2b^2/a = 32/3$. Sketch: draw the central rectangle with corners $(\pm 3, \pm 4)$, draw the two diagonal asymptotes through the origin and the corners, mark the vertices $(\pm 3, 0)$ and foci $(\pm 5, 0)$, and draw the right branch passing through $(3, 0)$ opening rightwards and hugging the asymptotes, and the mirror-image left branch through $(-3, 0)$; the curve never crosses the strip $-3 < x < 3$. A point check confirms the equation: at $x = 5$, the standard form gives $y^2/16 = 25/9 - 1 = 16/9$, so $y^2 = 256/9$ and $y = \pm 16/3 \approx \pm 5.33$; solving directly from the original equation gives the same result, since $16(25) - 9y^2 = 144$ yields $9y^2 = 256$. The points $(5, \pm 16/3)$ therefore lie on the curve beyond the focus, as the sketch suggests.

Table 1: Features of the hyperbola $x^2/9 - y^2/16 = 1$

Feature	Value
Centre	(0, 0)
a, b, c	$a = 3$, $b = 4$, $c = 5$ ($c^2 = a^2 + b^2$)
Vertices	(3, 0) and (−3, 0)
Foci	(5, 0) and (−5, 0)
Eccentricity	$e = 5/3 \approx 1.67$
Asymptotes	$y = (4/3)x$ and $y = -(4/3)x$
Transverse / conjugate axis lengths	6 / 8

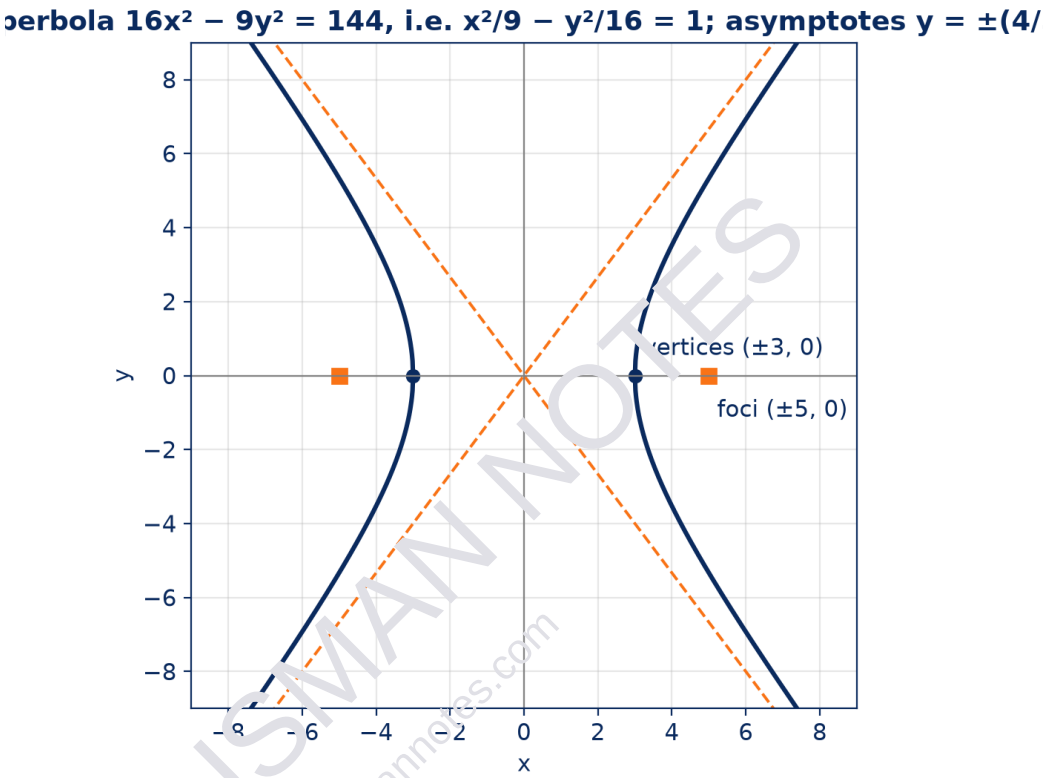


Figure 1: The hyperbola $16x^2 - 9y^2 = 144$ with its asymptotes, vertices and foci.

Part (b): Circle Through (1, 0), (0, 1) and (2, 2)

Three non-collinear points determine a unique circle (the points here are not collinear — the line through (1, 0) and (0, 1) is $x + y = 1$, and (2, 2) does not satisfy it). Use the general equation of a circle, $x^2 + y^2 + 2gx + 2fy + c_o = 0$, whose centre is $(-g, -f)$ and radius $\sqrt{(g^2 + f^2 - c_o)}$. Substituting (1, 0): $1 + 0 + 2g + 0 + c_o = 0$, so $2g + c_o = -1$. Substituting (0, 1): $0 + 1 + 0 + 2f + c_o = 0$, so $2f + c_o = -1$. Subtracting these two equations gives $g = f$, as the symmetry of the first two points about the line $y = x$ leads one to expect. Substituting (2, 2): $4 + 4 + 4g + 4f + c_o = 0$, so $4g + 4f + c_o = -8$; with $f = g$ this is $8g + c_o = -8$. Subtracting $2g + c_o = -1$ from $8g + c_o = -8$ gives $6g = -7$, so $g = -7/6$, and likewise $f = -7/6$, and then $c_o = -1 - 2g = -1 + 7/3 = 4/3$. The circle's equation is therefore $x^2 + y^2 - (7/3)x - (7/3)y + 4/3 = 0$, or in centre–radius form, by completing

the square, $(x - 7/6)^2 + (y - 7/6)^2 = 25/18$. Centre: $(-g, -f) = (7/6, 7/6) \approx (1.17, 1.17)$. Radius: $\sqrt{(g^2 + f^2 - c_o)} = \sqrt{(49/36 + 49/36 - 4/3)} = \sqrt{(98/36 - 48/36)} = \sqrt{(50/36)} = (5\sqrt{2})/6 \approx 1.18$. Verification against all three given points using the centre form: distance from (7/6, 7/6) to (1, 0) is $\sqrt{((1/6)^2 + (7/6)^2)} = \sqrt{(50/36)}$ ✓; to (0, 1) is $\sqrt{((7/6)^2 + (1/6)^2)}$, the same value ✓; to (2, 2) is $\sqrt{((5/6)^2 + (5/6)^2)} = \sqrt{(50/36)}$ ✓. All three distances equal the radius, and substitution into the general form also checks: for (2, 2), $4 + 4 - 14/3 - 14/3 + 4/3 = 8 - 24/3 = 0$ ✓. The centre (7/6, 7/6) lying on the line $y = x$ is consistent with the perpendicular bisector of the chord joining (1, 0) and (0, 1) being that line.

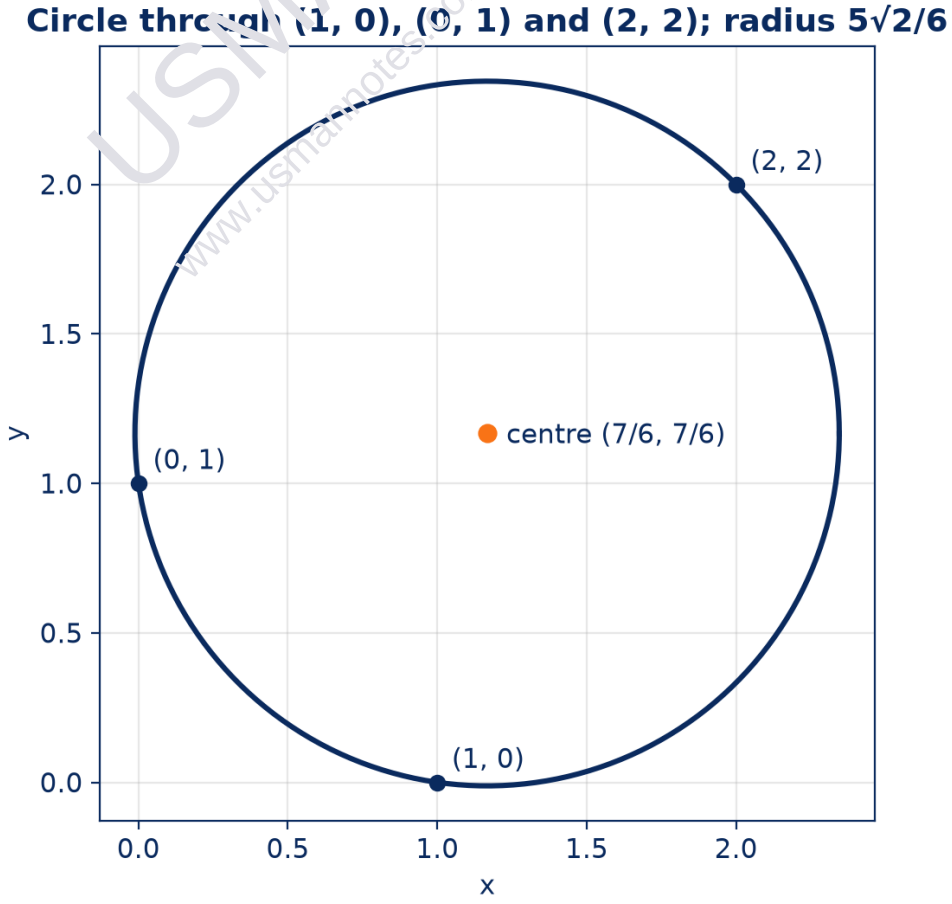


Figure 2: The circle through (1, 0), (0, 1) and (2, 2), centre (7/6, 7/6), radius 5√2/6.

Conclusion

For the hyperbola $16x^2 - 9y^2 = 144$, i.e. $x^2/9 - y^2/16 = 1$: centre (0, 0), vertices (±3, 0), foci (±5, 0), eccentricity 5/3, and asymptotes $y = \pm(4/3)x$, with a = 3, b = 4, c = 5. For the circle: centre (7/6, 7/6), radius $5\sqrt{2}/6$, and equation $(x - 7/6)^2 + (y - 7/6)^2 = 25/18$, equivalently $x^2 + y^2 - (7/3)x - (7/3)y + 4/3 = 0$, verified by equal distances to all three given points. Both answers follow the unit's standard pattern: reduce to standard or general form, extract the geometric data, and verify against the original data before sketching.

Question 5: A Polar Cardioid and Elimination of a Parameter (10+10)

Question as printed: (a) Sketch the polar curve $r = 2 + 2\cos\theta$. Name the curve, find its maximum and minimum values of r, and state its symmetry. (b) Given the parametric equations $x = t^2 - 1$, $y = 2t$, eliminate the parameter to find the Cartesian equation. Identify the curve.

Introduction

Cartesian coordinates are not the only way to describe a curve, and this question presents the two main alternatives. Polar coordinates locate a point by its distance r from the pole and the angle θ of the ray through it; they are the natural language for curves with circular symmetry, and circles, limaçons and cardioids have far simpler polar than Cartesian equations. Parametric equations go further and describe x and y separately as functions of a third variable t, often time, which is how moving points and computer-drawn curves are handled. Part (a) asks for the standard analysis of the polar curve $r = 2 + 2\cos\theta$: identify the family, tabulate enough values to sketch it, find the extreme values of r, and prove its symmetry from the equation rather than

from the picture. Part (b) asks for the reverse translation, eliminating t between $x = t^2 - 1$ and $y = 2t$ to obtain a single Cartesian equation, and then recognising the curve and its features from that equation. In each case the identification must be justified by the algebra, not guessed from the shape alone.

Part (a): The Polar Curve r = 2 + 2cos θ Is a Cardioid

The equation has the form $r = a(1 + \cos\theta)$ with $a = 2$. Curves $r = a \pm b\cos\theta$ (or with sin) are called limaçons; the special case a = b produces the heart-shaped curve with a cusp at the pole known as a cardioid — here a = b = 2, so the curve is a cardioid, symmetric about the initial line and pointing in the positive x-direction. To sketch it, tabulate r against θ. At $\theta = 0$: $r = 2 + 2 = 4$, the point (4, 0) in Cartesian terms. At $\theta = 60°$ ($\pi/3$): $\cos\theta = 1/2$, $r = 3$. At $\theta = 90°$ ($\pi/2$): $\cos\theta = 0$, $r = 2$, the point (0, 2). At $\theta = 120°$ ($2\pi/3$): $\cos\theta = -1/2$, $r = 1$. At $\theta = 180°$ (π): $\cos\theta = -1$, $r = 0$ — the curve reaches the pole. Continuing into the lower half: at $\theta = 240°$ ($4\pi/3$): $r = 1$; at $\theta = 270°$ ($3\pi/2$): $r = 2$, the point (0, −2); at $\theta = 300°$ ($5\pi/3$): $r = 3$; and at $\theta = 360°$: $r = 4$ again. Plotting these points and joining them smoothly gives the cardioid: a rounded heart shape, dimpled to a cusp exactly at the pole, bulging out to (4, 0) on the right, with topmost extent 2 and bottommost −2 in y. Maximum and minimum of r: since $\cos\theta$ ranges over $[-1, 1]$, $r = 2 + 2\cos\theta$ ranges over $[0, 4]$; the maximum value of r is 4, attained at $\theta = 0$ (the point furthest from the pole), and the minimum value is 0, attained at $\theta = \pi$, where the curve passes through the pole and forms its cusp. Note that r never becomes negative for this curve — the whole cardioid is traced once as θ runs from 0 to 2π. Symmetry: replace θ by −θ; $\cos(-\theta) = \cos\theta$, so r is unchanged, which proves the curve is symmetric about the polar axis (the x-axis, $\theta = 0$). The table shows the same fact numerically: the values at θ and at $360° - \theta$ agree in pairs (at 60° and

300° both r = 3, at 120° and 240° both r = 1), so the lower half is the mirror image of the upper half, and only the upper half ever needs to be plotted. There is no symmetry about the vertical line θ = π/2 or about the pole, as the values at θ = 0 (r = 4) and θ = π (r = 0) plainly differ. For reference, the Cartesian form, obtained from $r^2 = 2r + 2r\cos\theta$ with $r^2 = x^2 + y^2$ and $r\cos\theta = x$, is $x^2 + y^2 = 2\sqrt{(x^2 + y^2)} + 2x$, or squared, $(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$ — a fourth-degree equation, which is exactly why the polar form is preferred.

Table 1: Values of r = 2 + 2cos θ

θ	0°	60°	90°	120°	180°	240°	270°	300°	360°
r	4	3	2	1	0	1	2	3	4

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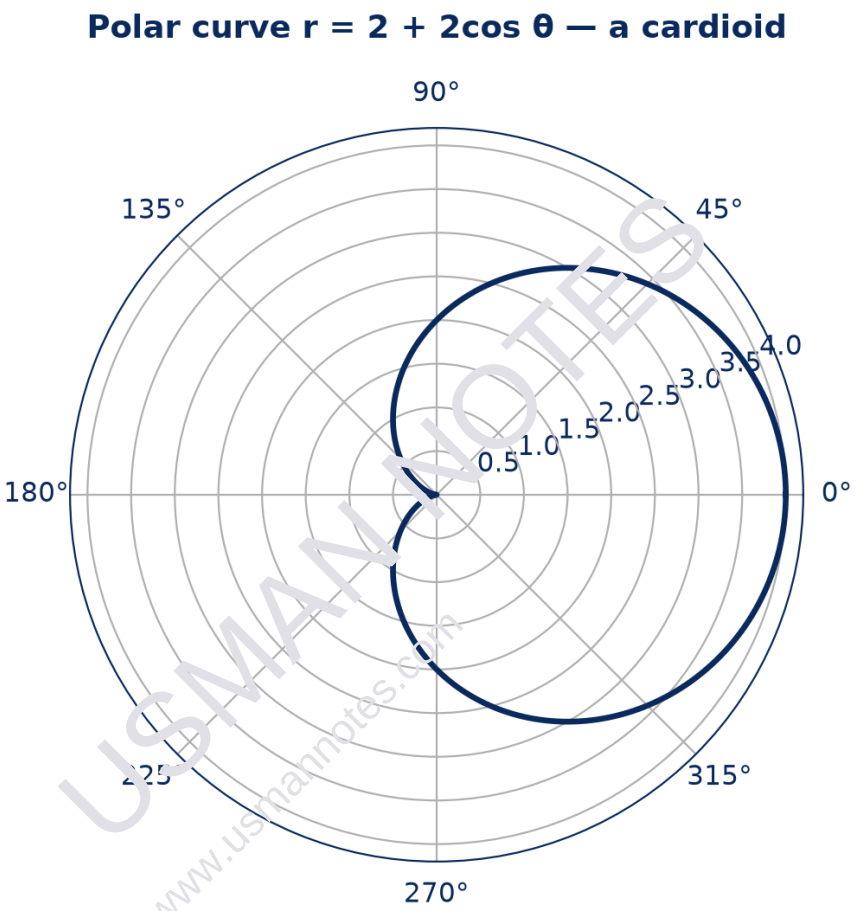


Figure 1: The cardioid r = 2 + 2cos ϑ, with maximum r = 4 at ϑ = 0 and cusp at the pole where r = 0.

Part (b): Eliminating t from $x = t^2 - 1$, $y = 2t$

The parameter t appears linearly in the y-equation, so solve there first: $y = 2t$ gives $t = y/2$. Substitute into the x-equation: $x = t^2 - 1 = (y/2)^2 - 1 = y^2/4 - 1$. Rearranging, $y^2 = 4(x + 1)$, or equivalently $y^2 = 4x + 4$. This is the Cartesian equation. Identify the curve: $y^2 = 4(x + 1)$ is of the standard parabola form $y^2 = 4a(x - h)$ with $a = 1$ and $h = -1$ (and $k = 0$), so the curve is a parabola with vertex at $(-1, 0)$, axis the x-axis (the line $y = 0$), opening to the right, focus at $(h + a, 0) = (0, 0)$ — remarkably, the focus is the origin — directrix the vertical line $x = h - a = -2$, and latus rectum of length $4a = 4$. The parametrisation traces the entire parabola exactly once as t

ranges over all real numbers: t = 0 gives the vertex (−1, 0); t = 1 gives (0, 2); t = −1 gives (0, −2), the two endpoints of the latus rectum through the focus; t = 2 gives (3, 4) and t = −2 gives (3, −4); as |t| grows, x grows like t^2 while y grows like 2t, so the arms recede to the right, symmetric about the x-axis because replacing t by −t leaves x unchanged and negates y. Check that these parametric points satisfy the Cartesian equation: for (3, 4), $y^2 = 16$ and $4(x + 1) = 16$ ✓; for (0, −2), $y^2 = 4$ and $4(0 + 1) = 4$ ✓. One caution is worth stating: elimination can in principle introduce restrictions (if, for example, t were confined to $t \geq 0$, only the upper half $y \geq 0$ would be traced); here t is unrestricted, and since $x = t^2 - 1 \geq -1$ matches exactly the natural domain of the parabola $y^2 = 4(x + 1)$, no part is lost and no part is added — the parametric and Cartesian descriptions are fully equivalent.

Table 2: Points of the parametric curve $x = t^2 - 1$, $y = 2t$

t	−2	−1	0	1	2
(x, y)	(3, −4)	(0, −2)	(−1, 0) vertex	(0, 2)	(3, 4)

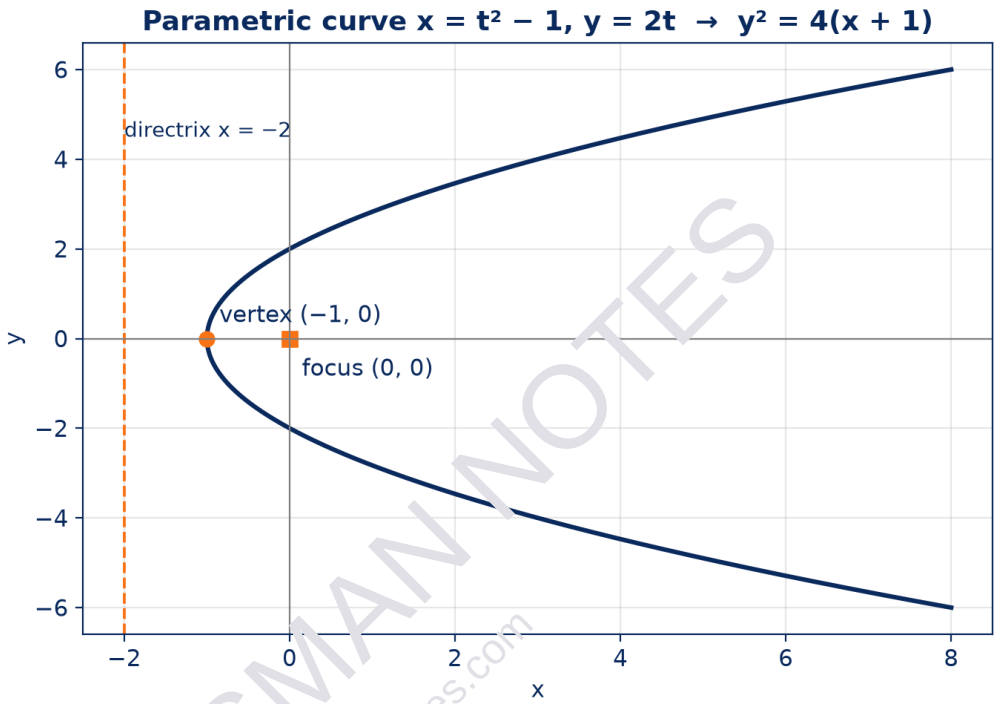


Figure 2: The parabola $y^2 = 4(x + 1)$ traced by $x = t^2 - 1$, $y = 2t$, with vertex (−1, 0), focus (0, 0) and directrix $x = -2$.

Conclusion

Part (a): $r = 2 + 2\cos\theta$ is a cardioid (the limaçon with a = b), with maximum $r = 4$ at $\theta = 0$, minimum $r = 0$ at $\theta = \pi$ where the cusp sits at the pole, and symmetry about the polar axis because cos is an even function. Part (b): eliminating t between $x = t^2 - 1$ and $y = 2t$ gives the Cartesian equation $y^2 = 4(x + 1)$, a parabola with vertex (−1, 0), focus (0, 0), directrix $x = -2$, opening to the right, and the parametrisation traces all of it. The two parts together demonstrate the unit's central skill: choosing the coordinate system — polar, parametric or Cartesian — in which a curve is simplest, and translating correctly between them.