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Name:

Registration No.:

Course Code: 3517

Program: BS

Semester: Autumn 2026

ALLAMA IQBAL OPEN UNIVERSITY, ISLAMABAD

(Department of Computer Science)

Course: Pre-Calculus I (3517) • Level: BS

Semester: Autumn 2026

Assignment No. 1 (Units 1-4)

Total Marks: 100 • Pass Marks: 50

SOLVED ASSIGNMENT GUIDE — For Study & Reference

Note: This guide is prepared for study and reference purposes only. Students are advised to read the relevant units of the course book, understand every definition, theorem and step of working, practise each type of question by hand, and then write the assignment in their own words and in their own handwriting or typing. Copying this guide word for word is not allowed. Every calculation in this guide has been checked by substitution or by an independent method, and students should repeat those checks themselves before submitting.

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Question 1: Periodicity and Differentiability of |sin x|, and Discontinuities of a Rational Function (10+10)

Question as printed: (a) Show that f(x) = |sin x| is periodic with period π but is not differentiable at x = nπ for any integer n. Sketch its graph. (b) Find all discontinuities of f(x) = 1/(x² − 3x + 2) and classify each as removable, jump, or infinite discontinuity.

Introduction

This question brings together three of the most basic ideas about how a function behaves: whether it repeats itself after a fixed interval, whether it has a well-defined tangent at every point, and whether its graph is broken at any point. Part (a) studies $f(x) = |\sin x|$, a function obtained by reflecting the negative humps of the sine curve above the x-axis. Because of that reflection the function repeats twice as fast as $\sin x$ itself, but the same reflection creates sharp corners exactly where the graph meets the x-axis, and a sharp corner is the classic situation in which a derivative fails to exist even though the function is perfectly continuous. Part (b) studies a rational function whose denominator vanishes at two points, and asks us to find every discontinuity and to classify it using the one-sided limits at each suspect point. The methods used here — testing $f(x + T) = f(x)$ for periodicity, comparing left-hand and right-hand derivatives for differentiability, and computing one-sided limits for classification — are the standard tools of Units 1–4 and they will be applied carefully, step by step.

Part (a)(i): $f(x) = |\sin x|$ Is Periodic with Period π

A function f is called periodic if there is a positive number T such that $f(x + T) = f(x)$ for every x in the domain; the smallest such T is called the (fundamental) period. We test the candidate $T = \pi$. Using the addition formula, $\sin(x + \pi) = \sin x \cos \pi + \cos x \sin \pi = -\sin x$, because $\cos \pi = -1$ and $\sin \pi = 0$. Taking absolute values gives $f(x + \pi) = |\sin(x + \pi)| = |-\sin x| = |\sin x| = f(x)$ for every real x. So π is certainly a period of f, exactly as claimed. It remains to see that no smaller positive number works, so that π is the fundamental period. Suppose $0 < T < \pi$ were a period. Then in particular $f(T) = f(0 + T) = f(0) = |\sin 0| = 0$, so $|\sin T| = 0$, which forces T to be a multiple

of π; but no multiple of π lies strictly between 0 and π. This contradiction shows that no smaller positive period exists. The reason the period is halved compared with sin x itself (whose period is 2π) is clear geometrically: the absolute value folds each negative half-wave of sine up into a positive hump identical to the positive half-wave, so the pattern of humps repeats after every interval of length π instead of every 2π.

Part (a)(ii): f Is Not Differentiable at x = nπ

The derivative at a point exists only if the left-hand and right-hand derivatives agree. Write f in pieces. On each interval (nπ, (n+1)π) the sign of sin x is constant — positive when n is even and negative when n is odd — so on such an interval f equals either sin x or −sin x and is smooth there, with f′(x) = cos x or f′(x) = −cos x respectively. The only doubtful points are the endpoints x = nπ, where the formula changes. It is enough to examine x = 0 in detail and then note that the same behaviour repeats at every multiple of π by periodicity. For small positive x, sin x > 0, so f(x) = sin x and the right-hand derivative at 0 is lim[h→0+] (sin h − 0)/h = 1, the standard limit. For small negative x, sin x < 0, so f(x) = −sin x and the left-hand derivative at 0 is lim[h→0−] (−sin h)/h = −1. Since −1 ≠ 1, the two one-sided derivatives at 0 differ, so f′(0) does not exist: the graph has a sharp corner (a cusp-like point or 'V' point) at the origin, with tangent slope −1 arriving from the left and slope +1 leaving to the right. Exactly the same computation at a general point x = nπ gives right-hand derivative +1 and left-hand derivative −1, because near nπ the function behaves like |sin(x − nπ)| up to sign, and the absolute value always produces the same corner. Hence f is not differentiable at x = nπ for any integer n, although f is continuous there — this example is the standard warning that continuity does not imply differentiability.

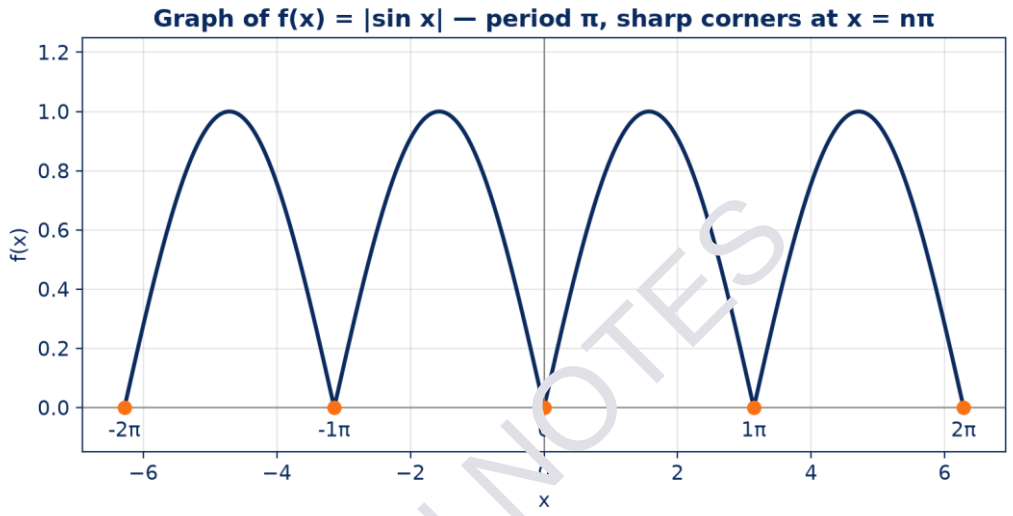


Figure 1: The graph of f(x) = |sin x|. Identical humps repeat with period π, and there is a sharp corner at each point x = nπ.

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Part (b): Discontinuities of f(x) = 1/(x² − 3x + 2)

A rational function is continuous at every point of its domain, so discontinuities can occur only where the function is undefined, that is, where the denominator is zero. Factorising, $x^2 - 3x + 2 = (x - 1)(x - 2)$, so the denominator vanishes at x = 1 and x = 2, and these are the only two candidate points. The numerator is the constant 1, which never vanishes, so no cancellation is possible and neither point can be a removable 'hole' caused by a common factor; we confirm this by the limits. At x = 1: as $x \to 1-$, the factor (x − 1) is a small negative number and (x − 2) is close to −1, so the denominator is a small positive number and $f(x) \to +\infty$; as $x \to 1+$, (x − 1) is small and positive while (x − 2) is still close to −1, so the denominator is small and negative and $f(x) \to -\infty$. The function therefore blows up on both sides of x = 1, and the line x = 1 is a vertical asymptote. At x = 2: as $x \to 2-$, (x − 1) is close to +1 and (x − 2) is small and negative, so $f(x) \to -\infty$; as $x \to 2+$, both factors are positive with (x − 2) small, so $f(x) \to +\infty$. Again the function

blows up, and x = 2 is a vertical asymptote. The classification is now immediate from the definitions. A removable discontinuity requires a finite two-sided limit that disagrees with, or replaces, the function value; here neither limit is finite. A jump discontinuity requires both one-sided limits to be finite but unequal; here the one-sided limits are infinite. An infinite discontinuity is precisely one in which at least one one-sided limit is +∞ or −∞. Therefore both x = 1 and x = 2 are infinite (non-removable) discontinuities, and f has no other discontinuities.

Table 1: One-sided limits and classification for f(x) = 1/((x − 1)(x − 2))

Point	Left-hand limit	Right-hand limit	Classification
x = 1	+∞	−∞	Infinite discontinuity; vertical asymptote x = 1
x = 2	−∞	+∞	Infinite discontinuity; vertical asymptote x = 2
All other real x	Equals f(x)	Equals f(x)	Continuous (no discontinuity)

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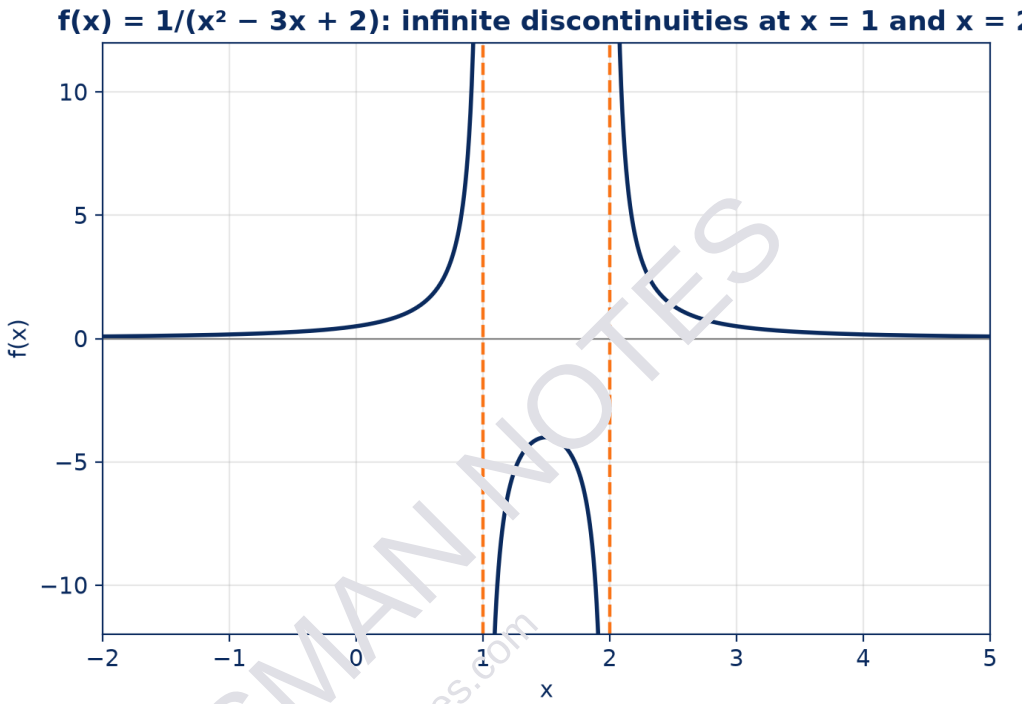


Figure 2: The graph of f(x) = 1/(x² − 3x + 2), showing the two vertical asymptotes.

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Conclusion

Part (a) shows the two faces of the absolute-value construction: f(x) = |sin x| satisfies f(x + π) = f(x) for all x, and π is its smallest period, yet at every point x = nπ the left-hand derivative −1 and the right-hand derivative +1 disagree, so the derivative does not exist and the graph has a sharp corner. Part (b) shows that the rational function 1/(x² − 3x + 2) is discontinuous only at x = 1 and x = 2, and that both discontinuities are infinite discontinuities with vertical asymptotes, neither removable nor jump. The general lessons are that classification of a discontinuity is always decided by the one-sided limits, and that a function can be continuous everywhere and still fail to be differentiable at isolated corner points.

Question 2: Remainder Theorem and Analysis of a Rational Function (10+10)

Question as printed: (a) Determine the remainder when $p(x) = x^5 - 6x^3 + 2x - 1$ is divided by $(x + 2)$. Factor $p(x)$ completely if $(x + 2)$ is a factor. (b) Analyze the rational function $f(x) = (x^2 - 1)/(x^2 - x - 6)$. Find all asymptotes, holes (if any), and sketch the graph.

Introduction

Part (a) is an application of two connected results about polynomials: the Remainder Theorem, which says that the remainder on dividing $p(x)$ by $(x - a)$ is simply $p(a)$, and the Factor Theorem, which says that $(x - a)$ is a factor exactly when that remainder is zero. These theorems turn a long-division problem into a single substitution, but they must be applied honestly: if the remainder is not zero, then the proposed factor is not a factor, and no factorisation through it exists — the correct answer is then the remainder together with the conclusion that the condition in the question is not fulfilled. Part (b) asks for a complete analysis of a rational function: factorising numerator and denominator, reading off the domain, holes, vertical, horizontal and oblique asymptotes and intercepts, and using a sign study to sketch the graph. Together the two parts test algebraic accuracy and the ability to draw correct conclusions from it.

Part (a): Remainder on Dividing $p(x) = x^5 - 6x^3 + 2x - 1$ by $(x + 2)$

The divisor is $(x + 2) = (x - (-2))$, so by the Remainder Theorem the remainder is $p(-2)$. Substituting carefully, term by term: $(-2)^5 = -32$; the x^4 term is missing, which means its

coefficient is 0; $-6x^3$ contributes $-6(-2)^3 = -6(-8) = +48$; the x^2 term is missing; $2x$ contributes $2(-2) = -4$; and the constant term is -1. Adding, $p(-2) = -32 + 48 - 4 - 1 = 11$. The remainder is therefore 11. As a check, synthetic division with root -2 on the full coefficient list 1, 0, −6, 0, 2, −1 (note the zeros inserted for the missing x^4 and x^2 powers, runs as follows: bring down 1; multiply by −2 to get −2 and add to 0 giving −2; multiply by −2 to get 4 and add to −6 giving −2; multiply by −2 to get 4 and add to 0 giving 4; multiply by −2 to get −8 and add to 2 giving −6; multiply by −2 to get 12 and add to −1 giving 11. The last entry, 11, agrees with $p(-2)$, and the quotient is $x^4 - 2x^3 - 2x^2 + 4x - 6$, so that $p(x) = (x + 2)(x^4 - 2x^3 - 2x^2 + 4x - 6) + 11$, which can be verified by expanding the right-hand side. Since the remainder 11 is not zero, the Factor Theorem tells us that $(x + 2)$ is NOT a factor of $p(x)$. The instruction 'factor $p(x)$ completely if $(x + 2)$ is a factor' is therefore conditional, and the condition fails: there is no factorisation of $p(x)$ with $(x + 2)$ as a factor, and it would be wrong to manufacture one. For completeness, the Rational Root Theorem adds that any rational root of $p(x)$ must be ± 1 (a factor of the constant term -1 divided by a factor of the leading coefficient 1); but $p(1) = 1 - 6 + 2 - 1 = -4$ and $p(-1) = -1 + 6 - 2 - 1 = 2$, so $p(x)$ has no rational linear factor at all, and in particular no complete factorisation into rational linear factors exists. The definite answers to part (a) are: remainder 11, and $(x + 2)$ is not a factor.

Table 1: Synthetic division of p(x) by (x + 2), root −2

Coefficients of p(x)	x^5: 1	x^4: 0	x^3: −6	x^2: 0	x^1: 2	Constant: −1
After synthetic	1	−2	−2	4	−6	Remainder = 11

division						

Part (b): Complete Analysis of f(x) = (x² − 1)/(x² − x − 6)

Step 1 — Factorise and find the domain. The numerator factorises as x² − 1 = (x − 1)(x + 1) and the denominator as x² − x − 6 = (x − 3)(x + 2). Hence f(x) = (x − 1)(x + 1)/((x − 3)(x + 2)), defined for all real x except x = 3 and x = −2, where the denominator is zero. Step 2 — Holes. A hole occurs at a value that makes both numerator and denominator zero, that is, at a common factor. Here the numerator vanishes at x = 1 and x = −1 while the denominator vanishes at x = 3 and x = −2; there is no common factor and no common zero, so the function has NO holes. Step 3 — Vertical asymptotes. At each excluded point where the numerator does not also vanish, the function blows up: x = 3 and x = −2 are vertical asymptotes. Step 4 — Horizontal asymptote. The degrees of numerator and denominator are equal (both 2), so the horizontal asymptote is the ratio of the leading coefficients, y = 1/1 = 1; indeed, dividing numerator and denominator by x² gives f(x) = (1 − 1/x²)/(1 − 1/x − 6/x²) → 1 as x → ±∞. Step 5 — Oblique asymptote. An oblique asymptote can occur only when the numerator degree exceeds the denominator degree by exactly one; the degrees are equal, so there is no oblique asymptote. Step 6 — Intercepts. Setting x = 0 gives f(0) = (−1)/(−6) = 1/6, so the y-intercept is (0, 1/6). Setting the numerator to zero gives the x-intercepts (−1, 0) and (1, 0). Step 7 — Sign and shape. The critical points −2, −1, 1, 3 divide the line into five intervals. Testing one point in each: f(−4) = 15/14 > 0, so the graph is above the x-axis and approaches y = 1 from above on the far left; f(−1.5) = (1.25)/(−2.25) < 0 between −2 and −1; f(0) = 1/6 > 0 between −1 and 1; f(2) = 3/(−4) < 0 between 1 and 3; and f(4) = 15/2 > 0 to the right of 3, approaching y = 1 from above. Combining

this with the asymptotes and intercepts produces the sketch: a left branch in $x < -2$ settling onto $y = 1$, a middle branch crossing the axis at -1 and 1 and passing through $(0, 1/6)$ while plunging and rising at the two asymptotes, and a right branch coming down from $+\infty$ at $x = 3$ and settling onto $y = 1$.

Table 2: Summary of the analysis of $f(x) = (x^2 - 1)/(x^2 - x - 6)$

Feature	Result
Domain	All real x with $x \neq 3$ and $x \neq -2$
Holes	None (no common factor of numerator and denominator)
Vertical asymptotes	$x = -2$ and $x = 3$
Horizontal asymptote	$y = 1$
Oblique asymptote	None
x-intercepts	$(-1, 0)$ and $(1, 0)$
y-intercept	$(0, 1/6)$

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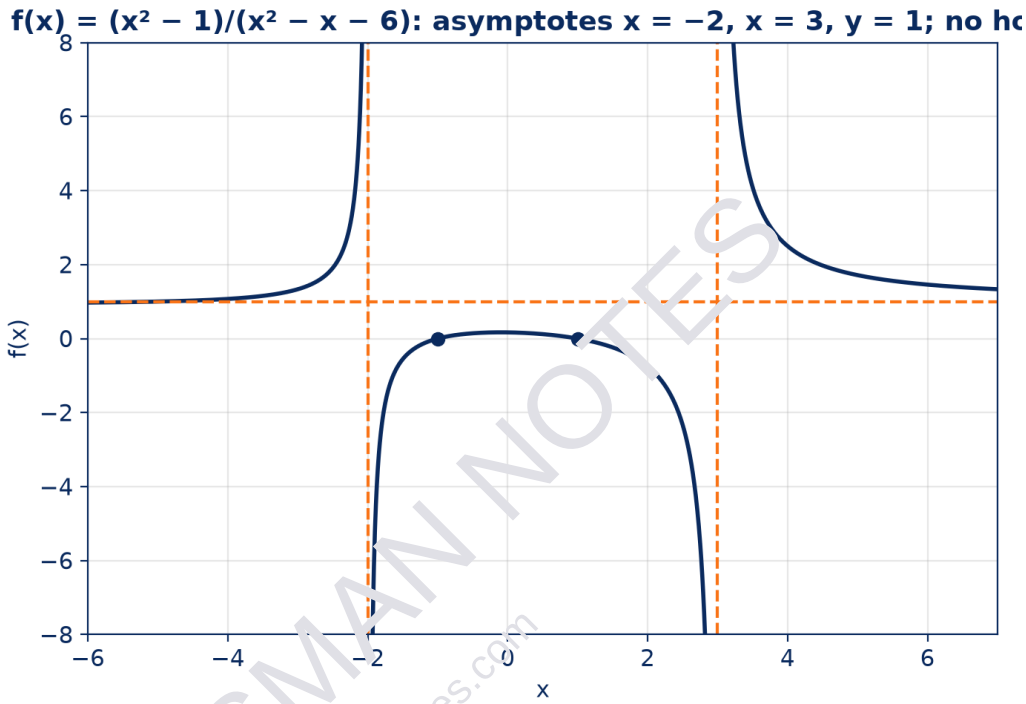


Figure 1: Graph of f(x) = (x² − 1)/(x² − x − 6) with its asymptotes x = −2, x = 3 and y = 1.

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Conclusion

In part (a) the Remainder Theorem gives the remainder 11 immediately, synthetic division confirms it, and because the remainder is not zero, (x + 2) is not a factor of p(x); the requested factorisation through (x + 2) therefore does not arise, and the Rational Root Theorem shows that p(x) in fact has no rational linear factor at all. In part (b) factorisation drives the whole analysis: the rational function has domain excluding −2 and 3, no holes, vertical asymptotes x = −2 and x = 3, horizontal asymptote y = 1, intercepts (−1, 0), (1, 0) and (0, 1/6), and a three-branched graph fully determined by these features together with a sign study. The common lesson is that factorisation, done first and done correctly, answers almost every subsequent question about a polynomial or rational function.

Question 3: Composition of Functions and Inverse of a Cubic Function (10+10)

Question as printed: (a) Let f(x) = x/(x − 2) and g(x) = 3x + 1. Find the domains of f ∘ g and g ∘ f. Are they equal? (b) Determine whether f(x) = x^3 + 2 is one-to-one. If so, find f^{-1}(x) and verify algebraically.

Introduction

Composition — applying one function to the output of another — is the main way new functions are built from old ones, and its subtle point is the domain: a value x can be used in f ∘ g only if g(x) exists AND the result g(x) lies in the domain of f. Students often find the formula for a composition correctly but report the domain wrongly by forgetting the second condition. Part (a) is designed to expose exactly that mistake, and also to show that composition is not commutative: f ∘ g and g ∘ f generally differ, in formula and in domain. Part (b) turns to the reverse process. A function has an inverse only when it is one-to-one, meaning that different inputs always give different outputs; for the cubic f(x) = x^3 + 2 we must first prove one-to-oneness, then construct the inverse by interchanging x and y and solving, and finally verify algebraically that the two functions undo each other. Both parts are foundational for the treatment of functions in Units 1–4.

Part (a): Domains of f ∘ g and g ∘ f

First note the individual domains: f(x) = x/(x − 2) is defined for all x ≠ 2, while the polynomial g(x) = 3x + 1 is defined for all real x. The composition (f ∘ g)(x) = f(g(x)) = f(3x + 1) = (3x + 1)/((3x

+ 1) − 2) = (3x + 1)/(3x − 1). For the domain, both conditions must hold: x must be in the domain of g (always true), and g(x) must be in the domain of f, that is, g(x) ≠ 2. Solving 3x + 1 ≠ 2 gives x ≠ 1/3. Hence the domain of f ∘ g is all real numbers except 1/3, written R \ {1/3} or (−∞, 1/3) ∪ (1/3, ∞). Notice how the exclusion moved: the forbidden input is no longer 2 but the input that g sends to 2. The other composition is (g ∘ f)(x) = g(f(x)) = 3·x/(x − 2)) + 1 = (3x + (x − 2))/(x − 2) = (4x − 2)/(x − 2). For its domain, x must be in the domain of f, so x ≠ 2, and f(x) must be in the domain of g, which imposes no restriction since g accepts every real number. Hence the domain of g ∘ f is R \ {2}. Are the two domains equal? No: R \ {1/3} and R \ {2} are different sets — for example 2 belongs to the domain of f ∘ g but not to that of g ∘ f, while 1/3 belongs to the domain of g ∘ f but not to that of f ∘ g. The functions themselves are also different, as one test value shows: at x = 0, (f ∘ g)(0) = (1)/(−1) = −1, whereas (g ∘ f)(0) = (−2)/(−2) = 1. So composition is not commutative here in either formula or domain, which is the expected general behaviour.

Table 1: The two compositions compared

Composition	Formula	Domain	Value at x = 0
f ∘ g	(3x + 1)/(3x − 1)	R \ {1/3}	−1
g ∘ f	(4x − 2)/(x − 2)	R \ {2}	1

Part (b): f(x) = x³ + 2 Is One-to-One, and Its Inverse

A function is one-to-one if f(a) = f(b) implies a = b. Suppose f(a) = f(b) for f(x) = x³ + 2. Then a³ + 2 = b³ + 2, so a³ = b³, so a³ − b³ = 0, and factorising, (a − b)(a² + ab + b²) = 0. We show the second factor cannot be zero unless a = b = 0 in a way that still forces a = b: indeed a² + ab + b² = (a + b/2)² + 3b²/4 ≥ 0, and it equals zero only when b = 0 and a = 0 simultaneously, in which case a =

b anyway. Hence in all cases a − b = 0, that is, a = b, and f is one-to-one. The same conclusion is visible graphically (the graph of a cubic with positive leading coefficient is strictly increasing and passes the horizontal line test) and analytically (x^3 is strictly increasing on R, and adding 2 merely shifts it up). Because f is one-to-one and its range is all of R (a cubic takes every real value), an inverse function f^{-1} with domain and range both equal to R exists. To find it, write $y = x^3 + 2$, interchange x and y to get $x = y^3 + 2$, and solve for y: $y^3 = x - 2$, so $y = \sqrt[3]{(x - 2)}$. Therefore $f^{-1}(x) = \sqrt[3]{(x - 2)}$, where $\sqrt[3]{}$ denotes the real cube root, which is defined for every real x — this is why, unlike the square root case, no domain restriction is needed. Verification is by composing in both orders. First, $(f \circ f^{-1})(x) = f(\sqrt[3]{(x - 2)}) = (\sqrt[3]{(x - 2)})^3 + 2 = (x - 2) + 2 = x$. Second, $(f^{-1} \circ f)(x) = f^{-1}(x^3 + 2) = \sqrt[3]{((x^3 + 2) - 2)} = \sqrt[3]{x^3} = x$. Both compositions give the identity function, which verifies algebraically that the two functions are genuine inverses. A numerical spot-check agrees: $f(1) = 3$ and $f^{-1}(3) = \sqrt[3]{1} = 1$; $f(-1) = 1$ and $f^{-1}(1) = \sqrt[3]{(-1)} = -1$.

Table 2: Matching values of f and f^{-1}

x	$f(x) = x^3 + 2$	Check: $f^{-1}(f(x))$
−1	1	$\sqrt[3]{(1 - 2)} = -1$ ✓
0	2	$\sqrt[3]{(2 - 2)} = 0$ ✓
1	3	$\sqrt[3]{(3 - 2)} = 1$ ✓
2	10	$\sqrt[3]{(10 - 2)} = 2$ ✓

Conclusion

In part (a), $(f \circ g)(x) = (3x + 1)/(3x - 1)$ has domain $R \setminus \{1/3\}$ while $(g \circ f)(x) = (4x - 2)/(x - 2)$ has domain $R \setminus \{2\}$; the domains are not equal, and the functions differ as well, confirming that

composition is order-dependent and that its domain is controlled by both functions, not just the outer formula. In part (b), $f(x) = x^3 + 2$ is proved one-to-one, its inverse is $f^{-1}(x) = \sqrt[3]{(x - 2)}$, and the two compositions $f \circ f^{-1}$ and $f^{-1} \circ f$ both reduce to x, which is the complete algebraic verification. The discipline illustrated in both parts — state the domain, prove the property, then verify by substitution — is exactly what further work with functions requires.

Question 4: The ε–δ Proof of a Limit and the Discontinuity of sin(1/x) (10+10)

Question as printed: (a) Using the ε-δ definition, prove that lim x→3 (2x − 1) = 5. (b) Identify and classify the discontinuities of f(x) = sin(1/x) at x = 0. Is the discontinuity removable? Explain.

Introduction

The ε–δ definition is the rigorous foundation of the limit concept: it turns the vague statement 'f(x) gets close to L as x gets close to a' into a challenge-and-response test that must work for every tolerance ε, however small. Part (a) applies this definition to the simplest non-trivial case, a linear function, where the proof can be seen in its purest form: the distance $|f(x) - L|$ turns out to be a constant multiple of $|x - a|$, so a suitable δ can be read off at once, and the structure — scratch work first, then the formal proof written forwards — is the template for every ε–δ argument. Part (b) examines the opposite extreme, a function whose behaviour near a point is so wild that no limit exists at all. The function sin(1/x) oscillates between −1 and 1 infinitely often in every neighbourhood of 0, and the task is to prove this with explicit

sequences and to classify the discontinuity correctly, in particular to explain precisely why it cannot be removed by defining or redefining a single value.

Part (a): ε–δ Proof that lim x→3 (2x − 1) = 5

First confirm the candidate limit value: substituting x = 3 gives 2(3) − 1 = 5, and since a polynomial is continuous, the limit is indeed expected to be 5; the task is to prove it from the definition. The definition states: lim x→a f(x) = L if for every ε > 0 there exists δ > 0 such that 0 < |x − a| < δ implies |f(x) − L| < ε. Scratch work (finding δ): compute |f(x) − 5| = |(2x − 1) − 5| = |2x − 6| = 2|x − 3|. We want this to be less than ε, and 2|x − 3| < ε exactly when |x − 3| < ε/2. This suggests the choice δ = ε/2. Formal proof: Let ε > 0 be given. Choose δ = ε/2, which is positive. Suppose 0 < |x − 3| < δ. Then |f(x) − 5| = |2x − 6| = 2|x − 3| < 2δ = 2(ε/2) = ε. Thus 0 < |x − 3| < δ implies |f(x) − 5| < ε, which is exactly what the definition requires. Therefore lim x→3 (2x − 1) = 5, proved. Three remarks complete the understanding. First, the condition 0 < |x − 3| excludes the point x = 3 itself: a limit concerns nearby values, not the value at the point (here they agree, but the definition never requires it). Second, δ depends on ε — halve the demanded accuracy and δ halves too — and any smaller positive δ would also work, since the implication only becomes easier to satisfy; δ is not unique. Third, the same pattern proves the general linear limit lim x→a (mx + b) = ma + b with δ = ε/|m| whenever m ≠ 0, so part (a) is a model for all linear functions, and geometrically it says that the points of the line y = 2x − 1 with x within δ of 3 all lie within the horizontal band of half-width ε around y = 5.

Table 1: Example ε–δ pairs for lim x→3 (2x − 1) = 5, with δ = ε/2

ε (allowed error in f(x))	δ = ε/2 (required closeness of x	Guarantee

	to 3)	
0.1	0.05	$\|x-3\|<0.05 \Rightarrow \|(2x-1)-5\|<0.1$
0.01	0.005	$\|x-3\|<0.005 \Rightarrow \|(2x-1)-5\|<0.01$
0.001	0.0005	$\|x-3\|<0.0005 \Rightarrow \|(2x-1)-5\|<0.001$

Part (b): The Discontinuity of f(x) = sin(1/x) at x = 0

The function f(x) = sin(1/x) is defined for all $x \neq 0$ and undefined at x = 0, so 0 is a discontinuity (the function is continuous at every other point). To classify it we investigate the limit as $x \to 0$ using sequences. Choose the sequence $x_n = 1/(2\pi n)$ for positive integers n. Then $x_n \to 0+$ and $f(x_n) = \sin(2\pi n) = 0$ for every n, so along this sequence f tends to 0. Choose instead $y_n = 1/(2\pi n + \pi/2)$. Then $y_n \to 0+$ and $f(y_n) = \sin(2\pi n + \pi/2) = 1$ for every n, so along this sequence f tends to 1. A third sequence, $z_n = 1/(2\pi n - \pi/2)$, gives $f(z_n) = -1$ throughout. Since the function approaches different values (0, 1 and −1) along different sequences tending to 0, the two-sided limit $\lim_{x\to 0} \sin(1/x)$ does not exist — indeed even the right-hand limit does not exist — and in fact f takes every value between −1 and 1 infinitely often in every interval $(0, \delta)$, because 1/x sweeps through infinitely many full periods of sine as x decreases to 0. Classification: the discontinuity is not a jump discontinuity, because a jump requires both one-sided limits to exist as finite numbers; it is not an infinite discontinuity, because f remains bounded between −1 and 1 and does not tend to $\pm\infty$ on either side; and it is not removable, as the next paragraph shows. In the

standard terminology it is an essential (oscillatory) discontinuity, sometimes called a discontinuity of the second kind. Is it removable? No. A discontinuity at a point is removable if the finite limit exists there, so that defining or redefining f at that single point by the limit value makes the function continuous. Here no finite limit exists – the values oscillate and come arbitrarily close to every number in [−1, 1] — so no choice whatsoever of a value f(0) can make the function continuous at 0: whatever value were assigned, points arbitrarily close to 0 would still carry function values far from it (for example values near 1 and near −1 both occur in every neighbourhood). The discontinuity is therefore non-removable in the strongest sense.

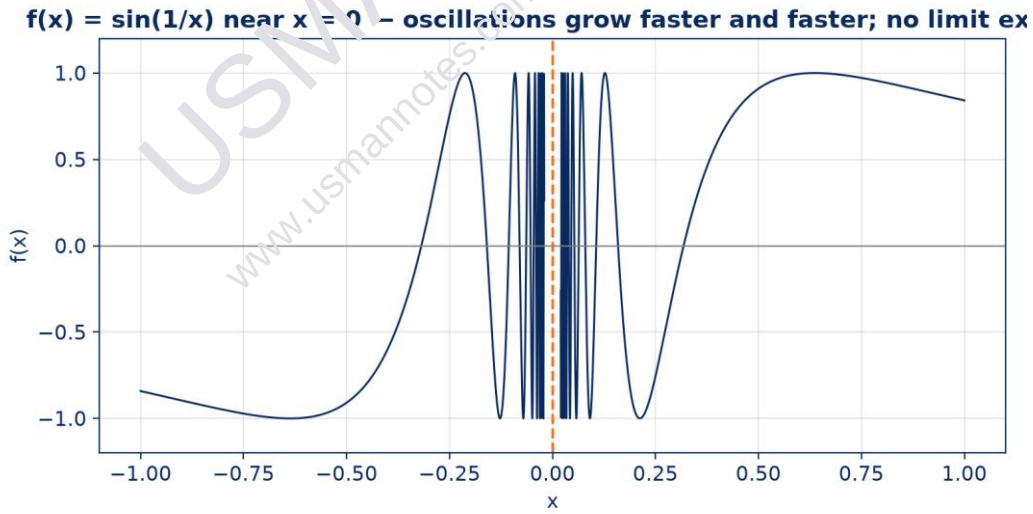


Figure 1: f(x) = sin(1/x) near 0. The oscillations become infinitely rapid and the limit does not exist.

Conclusion

Part (a) gives a complete ε–δ proof: with δ = ε/2, the implication $0 < |x - 3| < \delta \Rightarrow |(2x - 1) - 5| < \varepsilon$ holds because $|f(x) - 5| = 2|x - 3|$, so $\lim_{x \to 3} (2x - 1) = 5$ is established from the definition, not merely by substitution. Part (b) shows by explicit sequences that $\lim_{x \to 0} \sin(1/x)$ does not exist, so the discontinuity at 0 is an essential, oscillatory discontinuity — neither removable, nor

jump, nor infinite — and it cannot be removed by any definition of f(0). Placed side by side, the two parts mark the best and worst possible behaviour of a function near a point, and show why the ε–δ definition and the sequence criterion are the right tools for telling them apart.

Question 5: Domain, Range and Inverse of √(4 − x²), and a De Morgan Law (10+10)

Question as printed: (a) Find the domain, range, and inverse of f(x) = √(4 − x²). What geometric shape does the graph of f(x) represent? (b) If U = {1,2,3,4,5,6,7,8,9,10}, A = {1,2,4,8}, and B = {2,4,6,8,10}, verify (A ∪B)′ = A′ ∩ B′.

Introduction

Part (a) concerns the square-root function f(x) = √(4 − x²), and it contains an important lesson that must be handled honestly: the question asks for 'the inverse', but an inverse function exists only for a one-to-one function, and the first duty is to check that condition before computing anything. We shall find the domain and range directly, then test one-to-oneness, report the correct conclusion — that f as given has no inverse on its whole domain — and then give the standard remedy, restriction of the domain, under which an inverse does exist and can be written down and verified. Pretending that an inverse exists without the check would produce a formula that fails the defining test $f^{-1}(f(x)) = x$ on half the domain. The geometric question is answered by squaring, which reveals a circle, of which f is exactly half. Part (b) is a concrete verification of De Morgan's law for complements, one of the two laws that govern

how complement interacts with union and intersection; every set in the computation will be listed in full so that the equality can be checked element by element.

Part (a)(i): Domain and Range of f(x) = √(4 − x²)

Domain: a real square root requires the radicand to be non-negative, so we need $4 - x^2 \geq 0$, that is $x^2 \leq 4$, that is $-2 \leq x \leq 2$. The domain is the closed interval [−2, 2]. Range: on this domain, x^2 ranges from 0 (at x = 0) to 4 (at x = ±2), so $4 - x^2$ ranges from 4 down to 0, and its square root ranges from 2 down to 0. Every value in between is attained, by continuity, so the range is [0, 2]. The endpoints check out directly. f(0) = √4 = 2 is the maximum, and f(±2) = √0 = 0 is the minimum. These facts already describe the graph before any plotting: it lives entirely in the horizontal strip over [−2, 2], rises from (−2, 0) to a peak at (0, 2) and falls symmetrically to (2, 0), the symmetry being that of an even function, f(−x) = f(x).

Part (a)(ii): Does an Inverse Exist? One-to-One Test and Domain Restriction

Test for one-to-oneness: f(−1) = √(4 − 1) = √3 and f(1) = √(4 − 1) = √3, so two different inputs, −1 and 1, give the same output. Therefore f is NOT one-to-one on [−2, 2] (its graph fails the horizontal line test — the line y = √3 meets it twice), and consequently f, as given on its full domain, has NO inverse function. Any single formula claimed as 'the inverse' on [0, 2] would have to send √3 back to both −1 and 1 at once, which no function can do; for instance the naive candidate g(x) = √(4 − x²) gives g(f(−1)) = g(√3) = 1 ≠ −1, failing the verification. The standard remedy is to restrict the domain to a piece on which f is one-to-one. Restricting to [0, 2], the function is strictly decreasing there (from 2 to 0) and hence one-to-one, with range [0, 2]. Its inverse is found by setting y = √(4 − x²), swapping variables to get x = √(4 − y²), and solving: $x^2 =$

$4 - y^2$ with $y \geq 0$, so $y = \sqrt{(4 - x^2)}$. The inverse on the restricted domain is therefore $f^{-1}(x) = \sqrt{(4 - x^2)}$ with domain [0, 2] and range [0, 2] — the function is its own inverse (an involution) on [0, 2], and the verification $f^{-1}(f(x)) = \sqrt{(4 - (\sqrt{(4 - x^2)})^2)} = \sqrt{(x^2)} = x$ is valid precisely because $x \geq 0$ on the restricted domain. (On the alternative restriction [−2, 0] the inverse would instead be $f^{-1}(x) = -\sqrt{(4 - x^2)}$, domain [0, 2], range [−2, 0].) Students should reproduce this reasoning — test first, restrict if necessary, then invert — rather than reporting an unrestricted inverse, which does not exist.

Part (a)(iii): The Geometric Shape

Let $y = \sqrt{(4 - x^2)}$. Then $y \geq 0$ and squaring gives $y^2 = 4 - x^2$, that is $x^2 + y^2 = 4 = 2^2$, the equation of the circle centre (0, 0) and radius 2. The condition $y \geq 0$ selects exactly the upper half of that circle. Hence the graph of f is the upper semicircle of radius 2 centred at the origin, including its endpoints (−2, 0) and (2, 0) and its topmost point (0, 2). This identification explains everything found above at a glance: the domain [−2, 2] is the horizontal extent of the semicircle, the range [0, 2] is its vertical extent, and the failure of one-to-oneness is the obvious fact that most horizontal lines cut a semicircle in two points — the left half and the right half are mirror images in the y-axis, and restricting to [0, 2] keeps just the right-hand quarter-arc from (0, 2) to (2, 0), which is one-to-one.

Table 1: Values of $f(x) = \sqrt{(4 - x^2)}$ showing the symmetry $f(-x) = f(x)$

x	−2	−1	0	1	2
f(x)	0	$\sqrt{3} \approx 1.73$	2	$\sqrt{3} \approx 1.73$	0

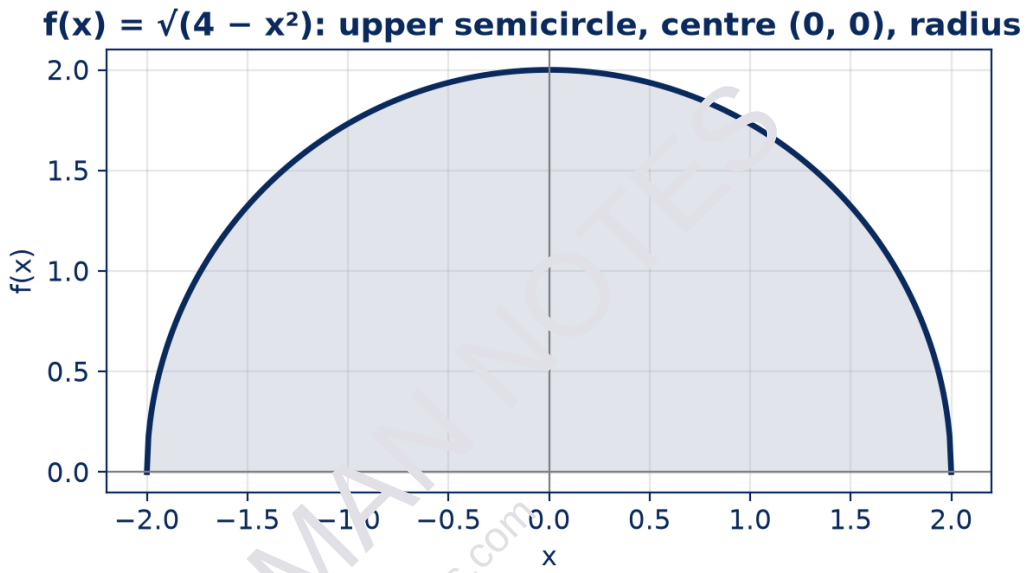


Figure 1: The graph of f(x) = √(4 − x²) is the upper semicircle of x² + y² = 4.

Part (b): Verification of (A ∪B)′ = A′ ∩ B′

Work with the roster method, writing every set out in full. Given U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, A = {1, 2, 4, 8} and B = {2, 4, 6, 8, 10}, where the prime denotes complement with respect to U. Left-hand side: first form the union, A ∪B = {1, 2, 4, 6, 8, 10} (every element that is in A or in B or in both, each written once). Its complement consists of the elements of U that are not in A ∪ B, namely (A ∪B)′ = {3, 5, 7, 9}. Right-hand side: the complement of A is A′ = {3, 5, 6, 7, 9, 10} (the elements of U missing from A), and the complement of B is B′ = {1, 3, 5, 7, 9} (the elements of U missing from B). Their intersection consists of the elements common to both complements: A′ ∩ B′ = {3, 5, 7, 9}. The two sides are identical sets, {3, 5, 7, 9} = {3, 5, 7, 9}, so De Morgan's law (A ∪B)′ = A′ ∩ B′ is verified for these sets. The verification also makes the

meaning of the law transparent: an element fails to be in A ∪B exactly when it is in neither A nor B, that is, when it lies in A′ and also in B′. Each of 3, 5, 7, 9 is indeed in neither A nor B, while every other element of U is in at least one of A, B and is correspondingly excluded from at least one complement — for example 6 is in B, so 6 ∉B′ and 6 ∉A′ ∩ B′, and 1 is in A, so 1 ∉A′ and 1 ∉ A′ ∩ B′.

Conclusion

In part (a), $f(x) = \sqrt{(4 - x^2)}$ has domain $[-2, 2]$ and range $[0, 2]$; it is not one-to-one on its full domain — $f(-1) = f(1)$ — so no unrestricted inverse exists, while on the restricted domain $[0, 2]$ it is one-to-one with inverse $f^{-1}(x) = \sqrt{(4 - x^2)}$ on domain and range $[0, 2]$; geometrically its graph is the upper semicircle of the circle $x^2 + y^2 = 4$. In part (b), direct listing gives $(A \cup B)' = \{3, 5, 7, 9\}$ and $A' \cap B' = \{3, 5, 7, 9\}$, so De Morgan's law is verified. The part (a) result is the one to remember for examinations: the words 'find the inverse' always carry the hidden first step of checking that an inverse can exist at all.