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Semester: Autumn 2026

ALLAMA IQBAL OPEN UNIVERSITY, ISLAMABAD

Course: Business Mathematics (1429) • Level: B.A, B.Com

Semester: Autumn 2026

Assignment No. 2 (Units 5–9)

SOLVED ASSIGNMENT GUIDE

(For Study & Reference)

Note: This guide is prepared for study and reference purposes. Students are advised to read the relevant units of the course book, understand the concepts and methods, and then write the assignment in their own words. Total marks of the assignment are 100 (pass marks 40); each question carries the marks shown with it. All working is shown step by step so that the method can be followed and reproduced.

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Question 1: Matrices (20 Marks)

a) Find the transpose of each of the following matrices: (i) A with rows (−1, 0, 0), (6, 9, −2), (−4, 5, 1); (ii) B with rows (6, −6), (4, 9), (−2, 0). b) If A is the 3 × 3 matrix with rows (1, 2, 3), (−1, −2, 3), (3, 4, 5), find 4A − 3B. c) The quarterly sales of wheat, cotton, and corn for the years 2010 and 2011 are represented below in the form of matrices A and B. Find the total quarterly sales of wheat, cotton, and corn for these two years.

Note on the printed question. Two printing notes must be stated honestly before the solutions, because a study guide must never invent data. In part (b) the booklet prints matrix A only; the matrix B needed for 4A − 3B is not printed anywhere in the booklet (and the B of part (a)(ii) is 3 × 2, so it cannot be subtracted from the 3 × 3 matrix A — the two do not even have the same

order). Part (b) is therefore solved as far as the printed data allow: 4A is computed in full, the entry-by-entry rule for subtracting 3B is stated, and a clearly labelled illustrative completion shows the final step. In part (c) the booklet says the sales matrices are 'represented below', but no matrices A and B are in fact printed below the question. Part (c) is therefore solved by the correct method, with a fully worked example on illustrative figures that are clearly marked as illustrative; the moment a student has the genuine A and B, the identical addition gives the answer.

Introduction

A matrix is a rectangular array of numbers arranged in rows and columns, and it is the natural way to store business data: sales of several products over several periods, costs of several inputs at several plants, or marks of several students in several subjects all sit comfortably in one matrix, with each position carrying a meaning fixed by its row and column. Once data are stored in matrices, whole tables can be processed in single operations — added when periods are combined, multiplied by a scalar when every figure is scaled, transposed when rows and columns exchange roles. This question exercises those three elementary operations. Part (a) asks for transposes, the operation of turning rows into columns. Part (b) asks for a linear combination, 4A − 3B, which combines scalar multiplication with subtraction. Part (c) is the addition operation wearing a business costume: two years of quarterly crop sales are to be combined into a single table of totals. None of these operations is difficult, but all of them are unforgiving of carelessness, because a single entry copied into the wrong position spoils a whole table silently. The defences are the ones used throughout this guide: state the order of every matrix before operating on it, work entry by entry with positions named, and check the

order of the result, which must obey simple rules — addition and subtraction require identical orders and preserve the order, scalar multiplication preserves the order, and transposition interchanges the order.

Part (a): Transposes

The transpose of a matrix A, written A^T, is obtained by writing the rows of A as the columns of A^T: the entry that stood in row i and column j moves to row j and column i. Consequently a matrix of order m × n has a transpose of order n × m, and the main diagonal entries stay where they are in a square matrix. (i) A is the 3 × 3 matrix with rows (−1, 0, 0), (6, 9, −2) and (−4, 5, 1). Its first row (−1, 0, 0) becomes the first column of A^T, its second row (6, 9, −2) becomes the second column, and its third row (−4, 5, 1) becomes the third column. The transpose is therefore the matrix with rows (−1, 6, −4), (0, 9, 5) and (0, −2, 1), printed in full below. A quick check: the diagonal entries −1, 9, 1 are unchanged, exactly as the rule promises, and transposing once more would restore A exactly. (ii) B is a 3 × 2 matrix, with rows (6, −6), (4, 9) and (−2, 0), so B^T must be 2 × 3 — the change of order is the whole point of this example. The rows of B become the columns of B^T, and B^T is the matrix with rows (6, 4, −2) and (−6, 9, 0). Students should verify each entry by its coordinates: for example the entry −2 stands in row 3, column 1 of B, so in B^T it stands in row 1, column 3 — and indeed −2 is the third entry of the first row of B^T. This position check, applied to two or three entries, is a complete safeguard against the commonest transposition error, which is copying a row across when it should have gone down.

Matrix A (as printed)

-1	0	0
6	9	-2
-4	5	1

Transpose A^T — answer to (a)(i)

-1	6	-4
0	9	5
0	-2	1

Matrix B (as printed)

6	-6
4	9
-2	0

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Transpose B^T — answer to (a)(ii)

6	4	-2
-6	9	0

Part (b): The Combination 4A − 3B

Scalar multiplication multiplies every entry of a matrix by the same number, and subtraction of matrices is performed entry by entry between matrices of the same order. The printed matrix is A with rows (1, 2, 3), (−1, −2, 3) and (3, 4, 5). Multiplying every entry by 4 gives 4A with rows (4, 8, 12), (−4, −8, 12) and (12, 16, 20), as shown below. If the missing matrix B has entries b_{ij}, then 3B has entries $3b_{ij}$, and 4A − 3B is the 3 × 3 matrix whose entry in row i and column j is $4a_{ij}$ −

$3b_{ij}$: that is, its first row is $(4 - 3b_{11}, 8 - 3b_{12}, 12 - 3b_{13})$, its second row is $(-4 - 3b_{21}, -8 - 3b_{22}, 12 - 3b_{23})$, and its third row is $(12 - 3b_{31}, 16 - 3b_{32}, 20 - 3b_{33})$. No numerical completion is possible until B is supplied, and any guide that printed one without comment would be fabricating the question. To make the method fully concrete, here is the completion for one illustrative B — the figures in this paragraph are illustrative only and do not come from the booklet. If, for example, B were the matrix with rows (1, 0, 2), (2, 1, 1) and (0, 3, 1), then 3B would have rows (3, 0, 6), (6, 3, 3) and (0, 9, 3), and subtracting entry by entry from 4A would give 4A − 3B with rows (1, 8, 6), (−10, −11, 9) and (12, 7, 17). The student's own B, when available, is substituted into exactly the same three steps: multiply A by 4, multiply B by 3, subtract corresponding entries. The frequent errors in such problems are subtracting before scaling, or worse, scaling only the first row; forming 4A and 3B as two separate complete matrices before touching the subtraction prevents both.

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Matrix A (as printed in part (b))

1	2	3
-1	-2	3
3	4	5

4A — the part of the answer fixed by the printed data

4	8	12
-4	-8	12
12	16	20

Part (c): Total Quarterly Sales for 2010 and 2011

The method the question is testing is matrix addition. The sales for 2010 form a matrix A and the sales for 2011 a matrix B. In the standard layout for such data, the three crops — wheat, cotton and corn — label the three rows, and the four quarters of the year label the four columns, so both A and B are 3 × 4 matrices and the entry in row 1, column 3, for example, is the wheat sales of the third quarter. The total sales over the two years are then given by the single addition A + B, in which every entry of the answer is the sum of the corresponding entries: $(\text{total})_{ij} = a_{ij} + b_{ij}$. Because the booklet does not print the two matrices (see the note at the head of this question), the following complete example uses illustrative figures, marked as such, in exactly the printed layout; the student's own figures, once supplied, go through the identical addition. Illustrative A (2010), in thousands of units: wheat row (40, 35, 50, 45), cotton row (30, 28, 33, 31), corn row (20, 22, 25, 24). Illustrative B (2011): wheat row (44, 38, 55, 48), cotton row (32, 30, 36, 34), corn row (21, 24, 27, 26). Adding quarter by quarter — wheat, first quarter: 40 + 44 = 84, and so on through all twelve positions — the total matrix A + B has wheat row (84, 73, 105, 93), cotton row (62, 58, 69, 65) and corn row (41, 46, 52, 50), as tabulated below. Reading the answer: total wheat sales were highest in the third quarter (105 thousand units in the illustration), and every crop's two-year total grew quarter by quarter in a pattern a manager can read straight off the combined table — which is precisely why businesses store such data as matrices in the first place. The checks are the usual ones: the answer must have the same order as the two given matrices (3 × 4), and spot-additions of individual entries must agree, for instance the corn figure for quarter 4, 24 + 26 = 50 ✓.

Table 1: Illustrative completion of part (c) — total sales matrix A + B (figures illustrative; method as printed)

Crop \ Quarter (illustrative totals, 000 units)	Q1	Q2	Q3	Q4
Wheat	84	73	105	93
Cotton	62	58	69	65
Corn	41	46	52	50

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Conclusion

To conclude: the transpose of the 3 × 3 matrix in (a)(i) is the matrix with rows (−1, 6, −4), (0, 9, 5), (0, −2, 1), and the transpose of the 3 × 2 matrix in (a)(ii) is the 2 × 3 matrix with rows (6, 4, −2), (−6, 9, 0). In part (b), 4A has rows (4, 8, 12), (−4, −8, 12), (12, 16, 20), and 4A − 3B is completed entry by entry as $4a_{ij} - 3b_{ij}$ once B is known; the booklet prints no B for this part, and the illustrative completion above shows the remaining step without pretending to data that were not given. In part (c), total two-year sales are the entry-by-entry sum A + B of the two yearly matrices, with the worked illustrative table showing the layout and the addition in full. The three operations share one discipline: respect the order of the matrices, since transposition interchanges it, scalar multiplication preserves it, and addition and subtraction are meaningless without it — and name the row and column of every entry while working, so that no figure can silently wander into the wrong cell.

Question 2: Determinants and Inverses of Matrices (20 Marks)

a) Find the determinant of the 4 × 4 matrix A with rows (1, 2, −1, 0), (0, 1, 2, 2), (1, 2, −1, 0), (2, −1, 1, 0). a) [numbered 'a)' again in the booklet] Evaluate the following determinants: (i) the 3 × 3 determinant with rows (2, 3, −1), (1, 1, 0), (2, −3, 5); (ii) the 3 × 3 determinant with rows (2a, a, a), (b, 2b, b), (c, c, 2c). b) Find A^{-1} if A is the 2 × 2 matrix with rows (5, 7), (9, 13).

Introduction

Every square matrix has a single number attached to it called its determinant, and that one number carries remarkable information: it tells whether the matrix can be inverted at all (it can, exactly when the determinant is not zero), it will reappear in the denominator when the inverse is computed, and it measures how the transformation represented by the matrix scales areas and volumes. Determinants also reward observation before computation. Certain patterns announce their answer in advance — a row or column of zeros, two identical rows or columns, or a triangular arrangement — and the skilled student scans for these patterns first, because they can turn a page of cofactor arithmetic into one line. This question is constructed to teach exactly that hierarchy of skills. The 4 × 4 determinant of the first part looks formidable but conceals a structural secret that settles it instantly, and finding the secret is plainly the point of the exercise. The two 3 × 3 determinants of the second part are honest computations, one numerical and one algebraic, the second of which also teaches the factoring properties of determinants. The final part asks for the inverse of a 2 × 2 matrix, where the determinant returns as the divisor and the answer can — and must — be verified by multiplying the matrix by its supposed inverse to recover the identity matrix. Throughout, the working follows the

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cofactor (Laplace) expansion: the determinant of an n × n matrix is built from determinants of size (n − 1) × (n − 1) with a chequerboard of + and − signs, and choosing the row or column with the most zeros minimises the labour.

Part (a): The 4 × 4 Determinant — Two Identical Rows

Before expanding anything, compare the rows of A. Row 1 is (1, 2, −1, 0) and row 3 is (1, 2, −1, 0): the two rows are identical, entry for entry. A fundamental property of determinants states that if two rows (or two columns) of a matrix are identical, its determinant is zero. The reason is not mysterious: interchanging the two identical rows must, by the sign-change property of determinants, change the sign of the determinant, yet the matrix after the swap is exactly the same matrix, so its determinant is unchanged; the only number equal to its own negative is 0. Therefore det A = 0 immediately, with no expansion at all. For completeness and practice, the same result can be seen by expansion along the fourth column, which is the natural choice because it contains two zeros: the entries of column 4 are 0, 2, 0, 0, so only the entry 2 in row 2 survives, with cofactor sign (−1)^(2+4) = +1. The determinant equals 2 times the 3 × 3 minor obtained by deleting row 2 and column 4, namely the determinant with rows (1, 2, −1), (1, 2, −1), (2, −1, 1) — and that minor itself has two identical rows, so it too is zero, confirming det A = 0 by direct computation as well. The consequence is worth noting: since det A = 0, the matrix A is singular and has no inverse; any system of equations with this coefficient matrix cannot have a unique solution. The lesson the examiner is teaching is that thirty seconds spent inspecting a determinant can be worth ten minutes of expansion, and that a zero determinant is information, not a failure — here it is detected twice over, by the identical rows of A and again inside its own minor.

Part (a), continued: Evaluating the Two 3 × 3 Determinants

(i) Expand the determinant with rows (2, 3, −1), (1, 1, 0), (2, −3, 5) along its first row, applying the sign pattern +, −, +. The first entry, 2, multiplies the minor obtained by deleting row 1 and column 1, which is (1 × 5 − 0 × (−3)) = 5, contributing 2 × 5 = 10. The second entry, 3, carries a minus sign and multiplies the minor (1 × 5 − 0 × 2) = 5, contributing −3 × 5 = −15. The third entry, −1, carries a plus sign and multiplies the minor (1 × (−3) − 1 × 2) = −3 − 2 = −5, contributing (−1) × (−5) = +5. The total is 10 − 15 + 5 = 0. The determinant is therefore 0. As with the 4 × 4 case, this zero is not a misprint to be distrusted: this matrix too is singular. A student who expected every exercise to produce a 'nice non-zero answer' might be tempted to fiddle a sign until a different number appeared; the correct response is to re-check the three minors — each is short — accept the verified zero, and note its meaning. (ii) The determinant with rows (2a, a, a), (b, 2b, b), (c, c, 2c) contains a common factor in each row: a can be taken out of the first row (every entry contains a), b out of the second row and c out of the third, using the property that multiplying a single row by a constant multiplies the determinant by that constant — here applied in reverse. The determinant therefore equals abc times the determinant with rows (2, 1, 1), (1, 2, 1), (1, 1, 2). Expanding that simpler determinant along the first row: 2 × (2 × 2 − 1 × 1) − 1 × (1 × 2 − 1 × 1) + 1 × (1 × 1 − 2 × 1) = 2 × 3 − 1 × 1 + 1 × (−1) = 6 − 1 − 1 = 4. Hence the required determinant is 4abc. A direct expansion without factoring gives the same result term by term — 2a(4bc − bc) − a(2bc − bc) + a(bc − 2bc) = 6abc − abc − abc = 4abc — and the agreement of the two routes is the check. The factoring route is the one to remember, because it converts an algebraic expansion full of products into a numerical one, and neat answers like 4abc are characteristic of determinants built on such patterns.

Part (b): The Inverse of A = [[5, 7], [9, 13]]

For a 2 × 2 matrix with entries a, b in the first row and c, d in the second, the inverse is given by A^{-1} = [1/(ad − bc)] × the matrix with rows (d, −b), (−c, a): the diagonal entries change places, the off-diagonal entries change sign, and the whole matrix is divided by the determinant. Here a = 5, b = 7, c = 9, d = 13, so the determinant is ad − bc = 5 × 13 − 7 × 9 = 65 − 63 = 2. The determinant is not zero, so the inverse exists — this check must come first, as Questions of this type occasionally hide a singular matrix for which the correct answer is 'no inverse exists'. Dividing by 2, the inverse is A^{-1} = (1/2) × the matrix with rows (13, −7), (−9, 5), that is, the matrix with rows (6.5, −3.5), (−4.5, 2.5). The verification is compulsory and conclusive: multiply A by A^{-1}. The (1,1) entry: 5 × 6.5 + 7 × (−4.5) = 32.5 − 31.5 = 1 ✓ The (1,2) entry: 5 × (−3.5) + 7 × 2.5 = −17.5 + 17.5 = 0 ✓ The (2,1) entry: 9 × 6.5 + 13 × (−4.5) = 58.5 − 58.5 = 0 ✓ The (2,2) entry: 9 × (−3.5) + 13 × 2.5 = −31.5 + 32.5 = 1 ✓ The product is the identity matrix, which is the definition of an inverse, so the answer is proved and not merely asserted. Note how small the determinant was — 2, the difference of 65 and 63 — and how a single slip (writing 5 × 13 = 75, for instance) would have poisoned every entry of the answer while leaving it looking plausible; the multiplication check exists precisely to catch that class of error, and an inverse presented without its check is unfinished work.

Conclusion

In summary: the 4 × 4 determinant is 0, because its first and third rows are identical (and the same fact reappears inside its cofactor minor); the first 3 × 3 determinant also evaluates to 0, by the honest expansion 10 − 15 + 5; the second 3 × 3 determinant evaluates to 4abc, most elegantly by factoring a, b and c out of the three rows and evaluating the remaining numerical

determinant as 4; and the inverse of the matrix with rows (5, 7), (9, 13) is (1/2) times the matrix with rows (13, −7), (−9, 5), verified by multiplication back to the identity matrix. The thread connecting all four parts is that determinants are governed by properties, not just procedures: identical rows force zero, row factors come out bodily, and the determinant stands guard over the inverse as its divisor and its existence test. A student who scans for structure first, expands carefully second, and verifies last will find this family of questions among the most reliable sources of marks in the whole paper.

Question 3: Differentiation (20 Marks)

a) Sketch the graph of the curve given by $f(x) = x^2 - 5x + 6$ and find its critical points. b) Find the derivative of the function $f(x) = x^5 - 3x^2 + 5x - 46$. c) The position of a particle moving along a straight line at time t is given by $s = f(t) = 4t^2 + 1$. Find the instantaneous rate of change at t = 2 seconds. d) The production costs per week for producing x widgets in a factory is given by $C(x) = 50 + 35x - 0.9x^2$. Calculate the cost to produce the 301st widget at x = 30 [as printed; see the note in part (d)].

Introduction

Differentiation measures rates of change and slopes. The derivative f′(x) of a function is itself a function which, at each point, gives the slope of the tangent to the graph and the instantaneous rate at which the quantity is changing. Almost everything in the applied half of this course flows from that single idea: the velocity of a moving particle is the derivative of its position, the marginal cost of production is the derivative of the total cost, and the peaks and troughs of a graph — the critical points — are found where the derivative is zero. This question visits all

three applications in turn. Part (a) combines curve sketching with the first critical-point calculation: a parabola, whose vertex is located by setting the derivative to zero. Part (b) is pure technique, the power rule applied term by term to a polynomial. Part (c) interprets the derivative physically, as the instantaneous velocity of a particle whose position law is given. Part (d) interprets the derivative economically, as marginal cost — the standard approximation to the cost of one more unit — and it also carries a cautionary tale, because the printed wording and the printed cost function do not sit comfortably together, and the honest treatment below says so and shows exactly what the printed numbers give. The power rule used repeatedly is this: the derivative of x^n is $n \cdot x^{n-1}$, the derivative of a constant multiple is the constant times the derivative, the derivative of a sum is the sum of the derivatives, and the derivative of a constant is zero.

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Part (a): Graph of $f(x) = x^2 - 5x + 6$ and Its Critical Points

The function $f(x) = x^2 - 5x + 6$ is a quadratic, so its graph is a parabola opening upwards (the coefficient of x^2 is positive). Three landmarks fix its shape. First, the roots: factorising, $x^2 - 5x + 6 = (x - 2)(x - 3)$, so $f(x) = 0$ at $x = 2$ and $x = 3$, and the curve crosses the x-axis at (2, 0) and (3, 0). Second, the y-intercept: $f(0) = 6$, so the curve crosses the y-axis at (0, 6). Third, the critical point. A critical point of a function is a point where the derivative is zero (or undefined); here $f'(x) = 2x - 5$, and setting $f'(x) = 0$ gives $2x = 5$, so $x = 2.5$ is the only critical value. The corresponding function value is $f(2.5) = (2.5)^2 - 5(2.5) + 6 = 6.25 - 12.5 + 6 = -0.25$, so the critical point is (2.5, −0.25). Since the parabola opens upwards, and since the second derivative $f''(x) = 2 > 0$ confirms it, this critical point is the vertex of the parabola and a minimum — the lowest point of the curve, lying, as symmetry demands, exactly midway between the roots 2 and 3, and dipping just

a quarter of a unit below the x-axis, which is why the two roots are so close together. A small table supports the sketch: f(0) = 6, f(1) = 2, f(2) = 0, f(2.5) = −0.25, f(3) = 0, f(4) = 2, f(5) = 6. The symmetry of the table about x = 2.5 (6, 2, 0, −0.25, 0, 2, 6) is the signature of the parabola and a built-in check on the arithmetic. The graph is drawn in Figure 1. For completeness: the function is decreasing on (−∞, 2.5) and increasing on (2.5, ∞), which is exactly what the sign of f′(x) = 2x − 5 says — negative before 2.5, positive after it.

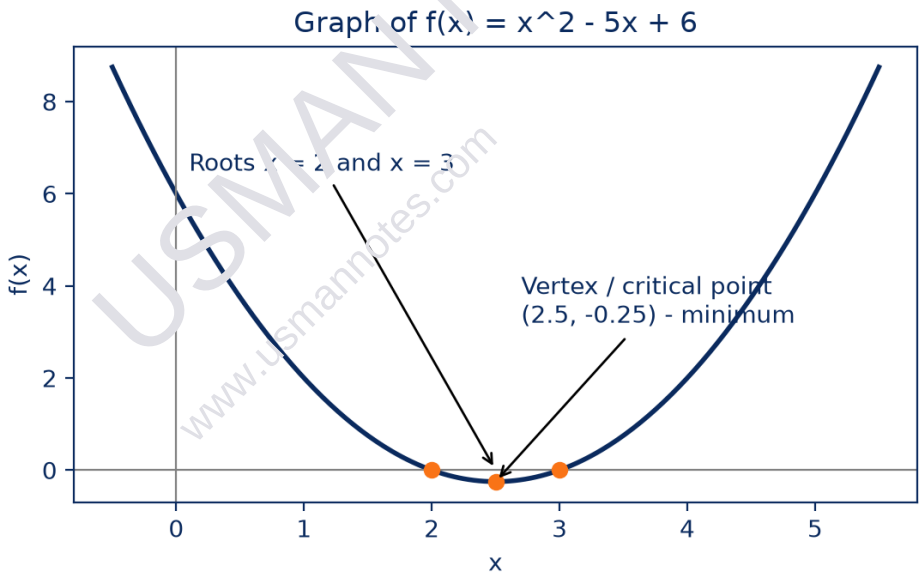


Figure 1: The parabola f(x) = x² − 5x + 6, with roots 2 and 3 and minimum (critical point) at (2.5, −0.25).

Part (b): Derivative of f(x) = $x^5 - 3x^{12} + 5x - 46$

Differentiate term by term with the power rule. The derivative of x^5 is $5x^4$. The derivative of $-3x^{12}$ is $-3 \times 12x^{11} = -36x^{11}$. The derivative of 5x is 5. The derivative of the constant −46 is 0. Collecting, $f'(x) = 5x^4 - 36x^{11} + 5$. The two traps in this routine exercise are worth naming. The first is the coefficient: the term is $-3x^{12}$, so the power 12 comes down and multiplies the existing coefficient −3, giving −36; students who merely write $-3x^{11}$ have applied only half of the

rule. The second is the exponent: each power falls by exactly one, so x^{12} gives x^{11} (not x^{12} still, and not x^{10}). The presence of a very high power, x^{12}, in a first exercise is unusual but changes nothing about the method — the power rule is the same for x^2, x^5 and x^{12} — and indeed the term $-36x^{11}$ dominates the behaviour of the derivative for large $|x|$. The result can be partially checked by reversing it mentally: integrating $5x^4$ returns x^5, integrating $-36x^{11}$ returns $-36 \times x^{12}/12 = -3x^{12}$, and integrating 5 returns 5x, while the constant −46 is invisible to differentiation — a reminder that the constant of a function can never be recovered from its derivative alone.

Part (c): Instantaneous Rate of Change of $s = 4t^2 + 1$ at $t = 2$

The position of the particle at time t is $s = f(t) = 4t^2 + 1$. The instantaneous rate of change of position — the velocity — is the derivative ds/dt. Differentiating, $ds/dt = 8t$ (the derivative of $4t^2$ is 8t, and the constant 1 vanishes). At the required instant t = 2 seconds, the instantaneous rate of change is $8 \times 2 = 16$ units of distance per second. To appreciate what 'instantaneous' adds, compare it with an average rate: between t = 2 and t = 3 the position moves from $s(2) = 4(4) + 1 = 17$ to $s(3) = 4(9) + 1 = 37$, an average rate of 20 per second; between t = 2 and t = 2.1 the average rate is $[4(2.1)^2 + 1 - 17]/0.1 = [17.64 + 1 - 17]/0.1 = 16.4$ per second; and as the interval shrinks towards zero the average rate closes in on 16, the derivative's value. The derivative is thus the limit of average rates, and 16 — not 17, the position, and not any particular average — is the speedometer reading at exactly t = 2. The accompanying observation is that the particle is speeding up: its velocity 8t grows steadily with time, its acceleration being the constant second derivative $d^2s/dt^2 = 8$. A complete answer therefore states the rate, its value at the required instant, and its units: the instantaneous rate of change of position at t = 2 seconds is 16 distance-units per second.

Part (d): Marginal Cost — the Cost of One More Widget

The cost of producing 'one more' unit is approximated in business mathematics by the marginal cost, the derivative C′(x) of the total cost function, evaluated at the current output: the extra unit is small compared with total output, so the rate of change of cost is an excellent estimate of the cost of the next unit, far quicker than computing C(x + 1) − C(x) exactly. Here C(x) = 50 + 35x − 0.9x², so the marginal cost function is C′(x) = 35 − 1.8x. Now the printed question must be handled carefully. As printed it asks for 'the cost to produce the 301st widget at x = 30', and those two numbers do not belong to the same situation: the 301st widget is the next unit when 300 have been produced (x = 300), while at x = 30 the next unit is the 31st. Both readings are therefore worked here, and the student can present the one matching the tutor's intention. Reading one — the printed x = 30: the marginal cost at x = 30 is C′(30) = 35 − 1.8 × 30 = 35 − 54 = −19, and the exact cost of the 31st widget agrees closely, C(31) − C(30) = 270.10 − 290 = −19.90. Reading two — the printed '301st' taken seriously, so x = 300: the marginal cost is C′(300) = 35 − 540 = −505, with exact value C(301) − C(300) = −70,955.90 − (−70,450) = −505.90. Both readings give a negative 'cost', and the negative sign is itself the most instructive part of the question. A negative marginal cost means total cost falls as output rises, which happens for this function because the −0.9x² term eventually overwhelms the +35x term: total cost here increases only up to x = 35/1.8 ≈ 19.4 widgets and decreases thereafter. Real cost functions do not behave like that over any wide range; the given function can sensibly describe this factory only for small outputs (roughly x below 19), and the exercise — a standard textbook one — is designed for the marginal-cost computation C′(x) = 35 − 1.8x and its evaluation, which the student should present exactly as above, noting the sign rather than concealing it. The method,

not the awkward numbers, is the mark-earning content: define marginal cost as the derivative, differentiate, evaluate at the stated output, and interpret the result.

Note on the printed question. The booklet's pairing of '301st widget' with 'x = 30' is internally inconsistent, as shown above. This guide follows the printed function exactly and reports both evaluations faithfully; nothing has been silently altered to force a tidier answer.

Conclusion

To conclude: the parabola $f(x) = x^2 - 5x + 6$ crosses the axes at (2, 0), (3, 0) and (0, 6) and has a single critical point, the minimum at (2.5, −0.25), where $f'(x) = 2x - 5$ vanishes. The derivative of $x^5 - 3x^{12} + 5x - 46$ is $5x^4 - 36x^{11} + 5$, by the power rule applied term by term. The particle with position $s = 4t^2 + 1$ has instantaneous rate of change 8t, which at t = 2 seconds equals 16 units per second. And the widget factory has marginal cost $C'(x) = 35 - 1.8x$, evaluating to −19 at the printed x = 30 (exact cost of the next unit −19.90) or −505 at x = 300 (exact −505.90), the negative sign exposing the limited range over which the printed cost function behaves realistically. One tool — the derivative — has served as slope-finder, velocity and marginal cost in a single question, which is precisely the unity the calculus units of this course are built to demonstrate.

Question 4: Partial Derivatives (20 Marks)

a) Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (4, −5) if $f(x, y) = 2x^2 + xy^2 + y - 1$. b) Find the second partial derivatives of the function f(x, y) = x^(2y) + y^3 [exponent as printed]. c) If $f(x, y) = 4x^2 + 2y^2 + xy$, find the critical points and determine if they are maxima or minima.

Introduction

A function of two variables, z = f(x, y), describes a surface rather than a curve: profit depending on the prices of two products, output depending on labour and capital, cost depending on the quantities of two goods. Its rates of change must therefore be taken one variable at a time. The partial derivative ∂f/∂x is the ordinary derivative with respect to x while y is held constant — the rate of change along a slice of the surface parallel to the x-direction — and ∂f/∂y is the mirror image with the roles exchanged. Part (a) of this question asks for both first partial derivatives of a polynomial function, evaluated at a specified point, which is the basic computational skill. Part (b) climbs one storey: the second partial derivatives f_{xx}, f_y_y and the mixed derivative f_x_y, which describe how the first rates of change themselves change, and which are the raw material of the maxima–minima test in Part (c). That final part applies the two-variable second-derivative test: at a critical point where both first partials vanish, the sign of the discriminant D = f_{xx} f_y_y − (f_x_y)², together with the sign of f_{xx}, decides whether the point is a local maximum, a local minimum or a saddle. The three parts therefore form one connected story — rates, rates of rates, and the classification of stationary points — and the same discriminant machinery returns in Question 5 for an optimisation problem. One notational comfort: for all well-behaved functions, including every function in this paper, the mixed derivatives are equal, f_x_y = f_y_x (Clairaut's theorem), so the mixed derivative need only be computed once, the second computation serving as a check.

Part (a): First Partial Derivatives of f(x, y) = 2x² + xy² + y − 1 at (4, −5)

To find ∂f/∂x, treat y as a constant and differentiate with respect to x. The term 2x² gives 4x. The term xy², in which y² is a constant multiplier, gives y². The term y is a constant as far as x is

concerned, so it gives 0, as does the constant −1. Hence ∂f/∂x = 4x + y². Evaluating at the point (4, −5): ∂f/∂x = 4(4) + (−5)² = 16 + 25 = 41. To find ∂f/∂y, treat x as a constant and differentiate with respect to y. The term 2x² is now constant and gives 0. The term xy² gives x × 2y = 2xy. The term y gives 1, and −1 gives 0. Hence ∂f/∂y = 2xy + 1. Evaluating at (4, −5): ∂f/∂y = 2(4)(−5) + 1 = −40 + 1 = −39. So at the point (4, −5), ∂f/∂x = 41 and ∂f/∂y = −39. The interpretation gives the numbers meaning: at that point on the surface, moving a small step in the positive x-direction increases f steeply, at a rate of about 41 units per unit step, while moving in the positive y-direction decreases f at a rate of about 39 per unit step; the surface is sloping strongly upward one way and downward the other, like a steep hillside traversed diagonally. Two routine slips should be guarded against. In ∂f/∂x, students sometimes differentiate y² as though y were x, producing a spurious 2y term; the golden rule is that while differentiating with respect to one variable, every appearance of the other variable is just a number. And in evaluation, (−5)² is +25, not −25 — the squaring of a negative ordinate is exactly where sign marks are lost in this standard exercise.

Part (b): Second Partial Derivatives of f(x, y) = x^(2y) + y³

As printed, the first term is x raised to the power 2y, where the exponent itself contains the variable y; the domain is x > 0, as real powers of this kind require. Such a term must be handled by the two faces of the power: with respect to x it behaves like a power function x^n with n = 2y, and with respect to y it behaves like an exponential, since x^(2y) = e^(2y ln x). The first partial derivatives are: ∂f/∂x = 2y · x^(2y − 1) (power rule, exponent 2y treated as constant), and ∂f/∂y = 2 ln x · x^(2y) + 3y² (exponential rule for the first term: multiply by the derivative of the exponent 2y ln x with respect to y, which is 2 ln x; ordinary power rule for y³). Differentiating

once more: $f_{xx} = \partial/\partial x\,[2y \cdot x^{(2y-1)}] = 2y(2y-1) \cdot x^{(2y-2)}$. Next, $f_y_y = \partial/\partial y\,[2 \ln x \cdot x^{(2y)} + 3y^2] = 2 \ln x \cdot [2 \ln x \cdot x^{(2y)}] + 6y = 4(\ln x)^2 \cdot x^{(2y)} + 6y$. Finally the mixed derivative, differentiating $\partial f/\partial x$ with respect to y (y now appears both as the leading factor and inside the exponent): $f_x_y = 2 \cdot x^{(2y-1)} + 2y \cdot x^{(2y-1)} \cdot (2 \ln x) = 2x^{(2y-1)}\,[1 + 2y \ln x]$. By Clairaut's theorem $f_y{}_x$ is the same expression, which may be confirmed by differentiating $\partial f/\partial y$ with respect to x — a worthwhile exercise, since agreement of the two mixed derivatives is the standard internal check on second-order work. The complete set is therefore: $f_x = 2y \cdot x^{(2y-1)}$; $f_y = 2(\ln x) \cdot x^{(2y)} + 3y^2$; $f_{xx} = 2y(2y-1)\,x^{(2y-2)}$; $f_y_y = 4(\ln x)^2 \cdot x^{(2y)} + 6y$; $f_x_y = f_y{}_x = 2x^{(2y-1)}[1 + 2y \ln x]$.

Remark on the reading of the term. If the printed term is read in the other typographically possible way, as $f(x, y) = x^y + y^3$ — the common textbook form of this exercise, in which the y is a multiplier rather than part of the exponent — the answers are correspondingly simpler, and they are recorded here so that the student is covered under either reading: $f_x = 2xy$, $f_y = x^2 + 3y^2$, $f_{xx} = 2y$, $f_y_y = 6y$, and $f_x_y = f_y{}_x = 2x$. The same three-step routine produces them: first partials by holding one variable constant, then differentiate each first partial once more, and check the mixed derivatives against each other.

Part (c): Critical Points of $f(x, y) = 4x^2 + 2y^2 + xy$

The first partial derivatives are $f_x = 8x + y$ and $f_y = 4y + x$. Critical points occur where both vanish simultaneously. From the first equation, $y = -8x$. Substituting into the second: $4(-8x) + x = 0$ gives $-32x + x = -31x = 0$, so $x = 0$, and then $y = -8 \times 0 = 0$. The function therefore has exactly one critical point, the origin (0, 0), at which $f(0, 0) = 0$. To classify it, compute the second partial

derivatives: $f_{xx} = 8$, f_y_y = 4, and f_x_y = 1 (constants, since the function is quadratic). The discriminant is $D = f_{xx} \cdot f_y_y - (f_x_y)^2 = 8 \times 4 - 1^2 = 32 - 1 = 31$. Since $D = 31 > 0$ and $f_{xx} = 8 > 0$, the second-derivative test declares (0, 0) a local minimum. More can honestly be said here: the minimum is in fact global, because the quadratic form $4x^2 + xy + 2y^2$ is positive for every $(x, y) \neq (0, 0)$ — for instance it may be written as $4(x + y/8)^2 + (31/16)y^2$, a sum of squares with positive coefficients, which is zero only at the origin — so $f(x, y) \geq 0$ everywhere and the least value of the function anywhere is $f(0, 0) = 0$. Geometrically the surface is an elliptic bowl whose rim rises in every direction, tilted slightly by the cross-term xy, with its bottom at the origin. The contrast with a saddle is instructive and previews Question 5: there the same test will return $D < 0$, and the stationary point will be a pass between hills rather than the bottom of a bowl — the sign of D, positive here at 31, is precisely what separates the two landscapes.

Conclusion

In conclusion: for $f(x, y) = 2x^2 + xy^2 + y - 1$, the partial derivatives at (4, −5) are $\partial f/\partial x = 41$ and $\partial f/\partial y = -39$. For f(x, y) = x^(2y) + y^3 as printed, the second partial derivatives are $f_{xx} = 2y(2y - 1)x^{(2y-2)}$, f_y_y = $4(\ln x)^2 x^{(2y)} + 6y$ and f_x_y = $f_{y_x} = 2x^{(2y-1)}[1 + 2y \ln x]$, with the alternative reading $x^2y + y^3$ — giving $f_{xx} = 2y$, f_y_y = 6y, f_x_y = 2x — also recorded. And $f(x, y) = 4x^2 + 2y^2 + xy$ has the single critical point (0, 0), which the second-derivative test ($D = 31 > 0$, $f_{xx} > 0$) identifies as a minimum, indeed the global minimum, with least value 0. The governing ideas are few and reusable: differentiate with respect to one variable while honestly treating the other as constant; evaluate carefully, especially squares of negative numbers; use the equality of the mixed derivatives as a check; and classify stationary points by the discriminant D

together with the sign of f_{xx} — the exact machinery that the optimisation question now puts to work.

Question 5: Optimisation (20 Marks)

a) Find the interval on which f is increasing, decreasing, concave up, and concave down for f(x) = $3x^3 - 2x - 15$. b) A company manufactures two types of a certain product. The joint cost function for producing x units of product A and y units of product B is given by $c = x^2 + 8xy + 10$. Find the quantity of each that results in the lowest cost. c) Locate the critical points and determine their nature for the function $f(x) = -2x^2 + (1/4)x^4$.

Introduction

Optimisation asks where a function is largest or smallest, and how it travels between those extremes. The analysis rests on two derivatives. The first derivative f′ controls direction: where f′ > 0 the function is increasing, where f′ < 0 it is decreasing, and where f′ = 0 the graph levels out at a critical point. The second derivative f″ controls bending: where f″ > 0 the graph is concave up (shaped like a cup), where f″ < 0 it is concave down (shaped like a cap), and the points where concavity changes are points of inflection. Part (a) of this question is a complete workout of that programme on a cubic. Part (b) transfers the idea to two variables and a business setting, a joint cost function of two products — and, taken exactly as printed, it contains a genuine surprise, examined honestly below: its only stationary point is a saddle, so there is no interior pair of quantities giving a lowest cost, and the correct professional answer is to demonstrate that, not to manufacture quantities the mathematics does not support. Part (c) returns to one variable with a quartic having three critical points of different characters, an

ideal test of the second-derivative test. Together the three parts exhaust the standard cases: monotone stretches separated by turning points, concavity separated by an inflection, a two-variable stationary point classified by the discriminant, and maxima and minima side by side on one curve. Sign charts are the organising tool throughout: factor the derivative, mark its zeros on a line, and test the sign in each interval — a routine that, once mastered, settles every question of this type.

Part (a): Increasing, Decreasing and Concavity of $f(x) = 3x^3 - 2x - 15$

Differentiate: $f'(x) = 9x^2 - 2$ and $f''(x) = 18x$. The critical values come from $f'(x) = 0$: $9x^2 = 2$, so $x^2 = 2/9$ and $x = \pm\sqrt{2}/3 \approx \pm 0.471$. Testing the sign of f' on the three intervals these values create: for $x < -0.471$, say $x = -1$, $f' = 9 - 2 = 7 > 0$, so f is increasing on $(-\infty, -\sqrt{2}/3)$; between the roots, say $x = 0$, $f' = -2 < 0$, so f is decreasing on $(-\sqrt{2}/3, \sqrt{2}/3)$; and for $x > 0.471$, say $x = 1$, $f' = 7 > 0$, so f is increasing on $(\sqrt{2}/3, \infty)$. Consequently $x = -\sqrt{2}/3$ carries a local maximum, with value $f(-0.471) = 3(-0.1047) + 0.943 - 15 \approx -14.37$, and $x = +\sqrt{2}/3$ carries a local minimum, with value $f(0.471) \approx -15.63$; the function rises to the first, falls to the second, and then rises for ever, as a cubic with positive leading coefficient must. For concavity, $f''(x) = 18x$ is negative when $x < 0$ and positive when $x > 0$, so the graph is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$, with a point of inflection at $x = 0$, where $f(0) = -15$: the curve passes through $(0, -15)$ changing its manner of bending there, cap-shaped on the left and cup-shaped on the right. The complete answer is therefore: increasing on $(-\infty, -0.471)$ and on $(0.471, \infty)$; decreasing on $(-0.471, 0.471)$; concave down on $(-\infty, 0)$; concave up on $(0, \infty)$; local maximum ≈ -14.37 at $x \approx -0.471$, local minimum ≈ -15.63 at $x \approx +0.471$, inflection at $(0, -15)$. Figure 1 draws the curve with these

features marked; note that the local maximum (−14.37) lies above the local minimum (−15.63), as it must, and that both are local only — the cubic itself is unbounded in both directions.

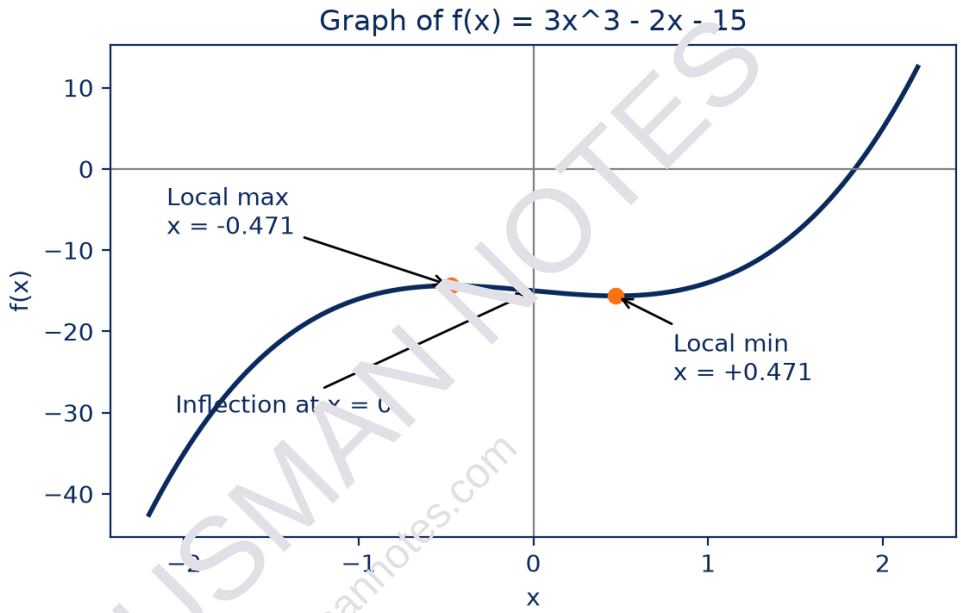


Figure 1: f(x) = 3x³ − 2x − 15: local maximum at x ≈ −0.471, local minimum at x ≈ +0.471, inflection at x = 0.

Part (b): The Joint Cost Function c = x² + 8xy + 10 — an Honest Analysis

The standard method is the one used in Question 4(c): find where both first partial derivatives vanish, then apply the second-derivative test. Here ∂c/∂x = 2x + 8y and ∂c/∂y = 8x (the derivative of x^2 with respect to y is 0, the derivative of 8xy with respect to y is 8x, and 10 is constant). Setting both to zero: from ∂c/∂y = 8x = 0 we get x = 0, and then ∂c/∂x = 2(0) + 8y = 0 gives y = 0. The only stationary point is (x, y) = (0, 0), at which c = 10. The second partial derivatives are c_{xx} = 2, c_y_y = 0 and c_x_y = 8, so the discriminant is D = c_{xx} c_y_y − (c_x_y)² = 2 × 0 − 8² = −64. Since D < 0, the stationary point is a saddle point — neither a maximum nor a minimum. The conclusion, stated plainly, is that the cost function as printed has no pair of

positive quantities (x, y) that yields a lowest cost: for positive x and y the cross-term 8xy pushes cost upward, while formally allowing negative y would drive the function down without bound, which is meaningless for quantities of a product. Restricted to the only sensible domain, $x \geq 0$ and $y \geq 0$, cost is smallest at the corner (0, 0) with c = 10 — that is, by producing nothing, which merely restates that this printed function, which rises with every unit produced and contains no revenue or demand side, poses no genuine production planning problem. This outcome should be reported, not disguised. Students meeting this question in the examination hall should reproduce exactly the analysis above — partial derivatives, the stationary point (0, 0), the discriminant $D = -64 < 0$, the verdict 'saddle point; no interior minimum exists' — because that is the mathematically correct answer to the question as printed, and it demonstrates the second-derivative test more convincingly than a routine minimum would. (Joint-cost exercises of this family normally contain terms in both x^2 and y^2 and linear terms, which do produce an interior minimum under the same method; the method itself is identical, and it is the printed function, not the method, that yields the saddle here.)

Part (c): Critical Points of $f(x) = -2x^2 + (1/4)x^4$ and Their Nature

Differentiate: $f'(x) = -4x + x^3$, which factorises conveniently as $f'(x) = x(x^2 - 4) = x(x - 2)(x + 2)$. The critical points are where $f' = 0$: $x = 0$, $x = 2$ and $x = -2$. The second derivative is $f''(x) = -4 + 3x^2$. Classify each point in turn. At $x = 0$: $f''(0) = -4 < 0$, so $x = 0$ is a local maximum, with value $f(0) = 0$. At $x = 2$: $f''(2) = -4 + 12 = 8 > 0$, so $x = 2$ is a local minimum, with value $f(2) = -2(4) + (1/4)(16) = -8 + 4 = -4$. At $x = -2$: $f''(-2) = -4 + 12 = 8 > 0$, so $x = -2$ is also a local minimum, with value $f(-2) = -8 + 4 = -4$, the same as at $x = 2$ — as expected, since f is an even function (it contains only even powers of x), its graph is symmetric about the y-axis and the two minima

must mirror each other in position and value. A sign chart for f′ confirms the classification from the direction of travel: f is increasing on (−2, 0) and (2, ∞), and decreasing on (−∞, −2) and (0, 2), so the graph descends into the minimum at (−2, −4), climbs to the maximum at (0, 0), descends again to the minimum at (2, −4), and then climbs without bound as the x^4 term takes over. Because the function tends to +∞ at both ends, the two local minima are also jointly the global minimum: the least value of f anywhere is −4, attained at two points. The local maximum value 0 at x = 0, by contrast, is exceeded for large |x|, a useful reminder that 'local' in these verdicts is a precise word. Figure 2 draws the characteristic W-shaped curve.

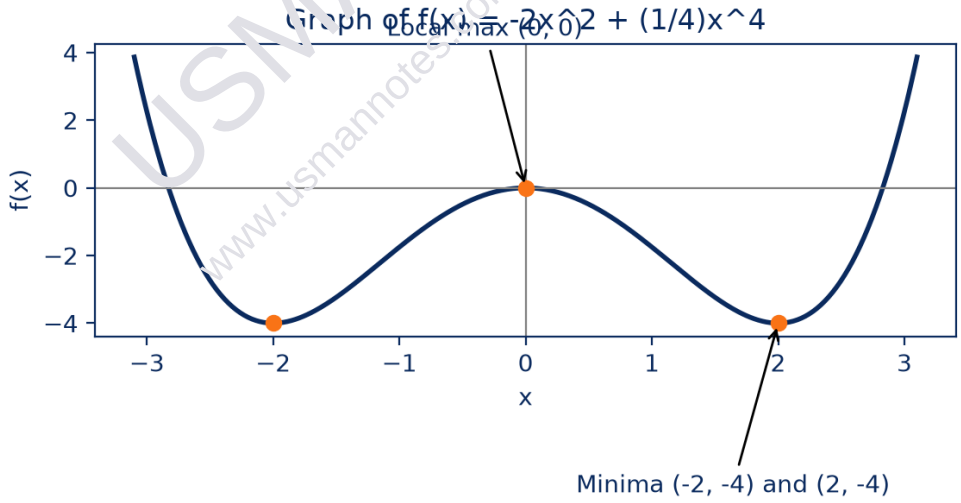


Figure 2: f(x) = −2x² + (1/4)x⁴, with local maximum (0, 0) and minima (−2, −4) and (2, −4).

Conclusion

To conclude: for f(x) = 3x³ − 2x − 15, the function is increasing on (−∞, −0.471) and (0.471, ∞), decreasing on (−0.471, 0.471), concave down on (−∞, 0) and concave up on (0, ∞), with a local

maximum of about −14.37 at x = −√2/3, a local minimum of about −15.63 at x = +√2/3, and an inflection point at (0, −15). For the joint cost function as printed, $c = x^2 + 8xy + 10$, the sole stationary point is (0, 0) and the discriminant D = −64 < 0 classifies it as a saddle, so no interior lowest-cost production pair exists — over non-negative quantities cost is minimised trivially at zero production, and the correct examination answer is the demonstration itself. For $f(x) = -2x^2 + (1/4)x^4$, the critical points are x = −2, 0 and 2: minima of −4 at both x = −2 and x = 2 (together the global minimum) and a local maximum of 0 at x = 0. The single routine behind all three parts deserves to be stated one final time: differentiate, solve f′ = 0 (or ∇f = 0 in two variables), and classify with f″ or the discriminant D, reporting honestly whatever the signs declare — including, as Part (b) shows, when they declare that the hoped-for optimum does not exist.

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