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ALLAMA IQBAL OPEN UNIVERSITY, ISLAMABAD

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Programme / Level: Intermediate (HSSC)

Department of Mathematics

Semester: Autumn 2026

Assignment No. 2 (Units 6–9)

Total Marks: 100 Pass Marks: 40

SOLVED ASSIGNMENT GUIDE — For Study & Reference

Note for students: This guide explains the method and the reasoning behind every step so that you can learn the topic properly. Please study the working, then write your assignment in your own words and in your own hand; copying any guide word-for-word is against the University's rules.

Question No. 1 [20] — (a) A circle is passing through the point (−1, 7) and a line 3x − 7y + 12 = 0 is tangent to the circle at (5, 2). Find the equation of the circle. (b) Write equations of two tangents from (1, 7) to the circle $3x^2 + 3y^2 = 15$.

Introduction

Part (a) asks for a circle determined by two points and a tangency condition. The governing principle is that the radius drawn to a point of tangency is perpendicular to the tangent line there; hence the centre must lie on the line through the point of contact (5, 2) perpendicular to the given tangent line, and it must be equidistant from the two known points (5, 2) and (−1, 7) of the circle. Those two conditions determine the centre uniquely. Before solving, however, this guide must flag a printing defect in the booklet, because checking data before using it is part of

good mathematical practice. Part (b) is the standard problem of the two tangents drawn from an external point to a circle, solved here by imposing that the distance from the centre to each candidate line equals the radius.

A note on the printed data of part (a) — read this first

TRANSPARENCY NOTE: substituting the alleged point of tangency (5, 2) into the printed line gives $3(5) - 7(2) + 12 = 15 - 14 + 12 = 13 \neq 0$, so the point (5, 2) does NOT lie on the line $3x - 7y + 12 = 0$ as printed; the constant term is inconsistent with the point (the constant would have to be −1, not +12, for the line to pass through (5, 2)). The direction of the line, however, is unambiguous, and the standard method uses only that direction: the centre lies on the normal through (5, 2), i.e. on the line through (5, 2) in the direction of the normal vector (3, −7). The solution below therefore proceeds by the standard method — centre on the normal at the point of contact, equidistant from (5, 2) and (−1, 7) — and the circle obtained has, at (5, 2), the tangent line $3x - 7y - 1 = 0$, which is parallel to the printed line and is evidently the line the booklet intended. Students should present the solution this way and may mention the misprint in one line.

Part (a): Locating the centre

Parametrize the normal line through (5, 2): the centre is $O = (5 + 3t, 2 - 7t)$ for some real t, since (3, −7) is perpendicular to the tangent direction. The radius squared is the squared distance from O to the point of contact (5, 2): $R^2 = (3t)^2 + (-7t)^2 = 58t^2$. The circle also passes through (−1, 7), so the squared distance from O to (−1, 7) must equal the same R^2: $(5 + 3t + 1)^2 + (2 - 7t - 7)^2 = (6 + 3t)^2 + (-5 - 7t)^2 = 58t^2 + 106t + 61$. Setting this equal to $58t^2$, the quadratic terms cancel

— this always happens, because the condition 'equidistant from two points' is linear — leaving $106t + 61 = 0$, so $t = -61/106$. The centre is therefore:

$$O = (5 + 3(-61/106), 2 - 7(-61/106)) = (347/106, 639/106) \approx (3.2736, 6.0283)$$

$$R^2 = 58(61/106)^2 = 107909/5618 \approx 19.2076, \quad R \approx 4.3827$$

Part (a) continued: The equation of the circle

In centre–radius form the circle is $(x - 347/106)^2 + (y - 639/106)^2 = 107909/5618$. Expanding gives the general form $x^2 + y^2 - (347/53)x - (639/53)y + 1476/53 = 0$, approximately $x^2 + y^2 - 6.547x - 12.057y + 27.849 = 0$. Verification against both given points is clean and worth showing: at (5, 2), $25 + 4 - (347/53)(5) - (639/53)(2) + 1476/53 = 29 + (-1735 - 1278 + 1476)/53 = 29 - 1537/53 = 29 - 29 = 0$ ✓(since $53 \times 29 = 1537$); at (−1, 7), $1 + 49 + 347/53 - (639/53)(7) + 1476/53 = 50 + (347 - 4473 + 1476)/53 = 50 - 2650/53 = 50 - 50 = 0$ ✓(since $53 \times 50 = 2650$). Tangency check: the radius to (5, 2) has slope $(639/106 - 2)/(347/106 - 5) = (427/106)/(-183/106) = -7/3$, so the tangent at (5, 2) has slope 3/7 and equation $3x - 7y - 1 = 0$ — parallel to the printed line $3x - 7y + 12 = 0$, sharing its normal direction (3, −7), exactly as the geometry requires.

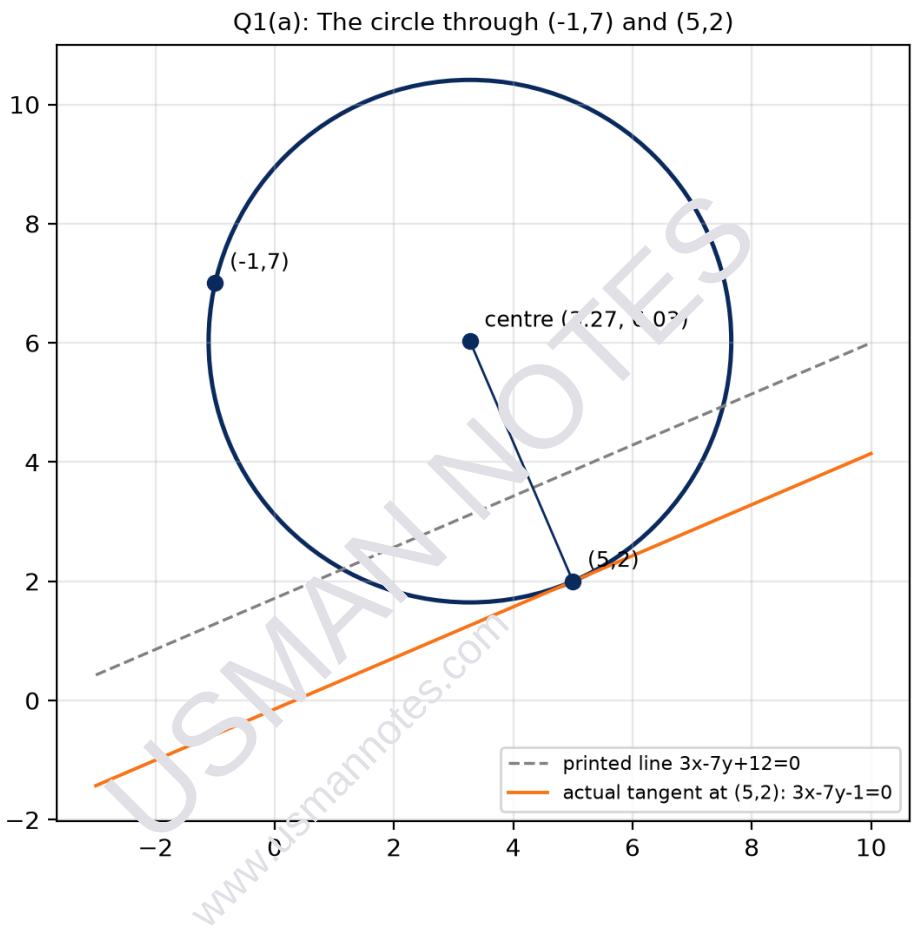


Figure: The circle of part (a), its centre, the two given points, the actual tangent at (5, 2), and the booklet's printed line (dashed, parallel to the tangent).

Part (b): Two tangents from (1, 7) to $3x^2 + 3y^2 = 15$

Dividing by 3, the circle is $x^2 + y^2 = 5$, with centre the origin O(0, 0) and radius $\sqrt{5} \approx 2.236$. The point P(1, 7) is outside the circle since $OP = \sqrt{(1 + 49)} = \sqrt{50} > \sqrt{5}$, so exactly two tangents can be drawn, as the question expects. Let a tangent through P have slope m: its equation is $y - 7 = m(x - 1)$, or $mx - y + (7 - m) = 0$. The tangency condition is that the perpendicular distance from the centre (0, 0) to this line equals the radius $\sqrt{5}$: $|7 - m|/\sqrt{(m^2 + 1)} = \sqrt{5}$. Squaring, $(7 - m)^2 = 5(m^2 + 1)$, i.e. $m^2 - 14m + 49 = 5m^2 + 5$, i.e. $4m^2 + 14m - 44 = 0$, or $2m^2 + 7m - 22 = 0$. Factoring, $(2m + 11)(m - 2) = 0$, so $m = 2$ or $m = -11/2$. (No vertical tangent is missed by the slope method:

the vertical line x = 1 is at distance 1 ≠ √5 from the origin, so it is not tangent.) The two tangents are therefore:

$$\text{Tangent 1 } (m = 2):\ y - 7 = 2(x - 1),\ \text{i.e. } 2x - y + 5 = 0$$

$$\text{Tangent 2 } (m = -11/2):\ y - 7 = -(11/2)(x - 1),\ \text{i.e. } 11x + 2y - 25 = 0$$

Verification: the distance from (0, 0) to 2x − y + 5 = 0 is |5|/√(4 + 1) = 5/√5 = √5 ✓, and to 11x + 2y − 25 = 0 is |−25|/√(121 + 4) = 25/√125 = 25/(5√5) = √5 ✓; both lines pass through (1, 7) by construction (2 − 7 + 5 = 0 and 11 + 14 − 25 = 0). The points of contact, found as the feet of the perpendiculars from the centre, are (−1, 1) for the first tangent and (11/5, 2/5) for the second; both lie on $x^2 + y^2 = 5$ (4 + 1 = 5 and 121/25 + 4/25 = 125/25 = 5), and both are at distance √(9 + 36) = √45 = 3√5 from (1, 7), agreeing with the tangent-length formula $\sqrt{(OP^2 - R^2)} = \sqrt{(50 - 5)} = \sqrt{45}$.

Conclusion

Part (a): notwithstanding the booklet's misprinted constant (the point (5, 2) does not satisfy 3x − 7y + 12 = 0; the intended tangent is 3x − 7y − 1 = 0), the standard normal-line method gives centre (347/106, 639/106) ≈ (3.27, 6.03), radius ≈ 4.38, and circle $x^2 + y^2 - (347/53)x - (639/53)y + 1476/53 = 0$, verified exactly at both given points. Part (b): the two tangents from (1, 7) to $x^2 + y^2 = 5$ are 2x − y + 5 = 0 and 11x + 2y − 25 = 0, each verified by the distance-equals-radius test.

Question No. 2 [20] — (a) Write an equation of the parabola with focus (−2, 5) and vertex (−2, 7). (b) Analyze equation $3x^2 - 5y^2 = 15$.

Introduction

Part (a) is a parabola determined by its focus and vertex. The vertex is the midpoint between the focus and the directrix, the axis is the line through the focus and vertex, and the signed distance a from vertex to focus fixes the standard equation completely, including the direction in which the parabola opens. Part (b) asks for a full analysis of a second-degree equation: the word 'analyze' in conic sections means to reduce the equation to standard form and then read off and report all the geometric elements — centre, axes, vertices, foci, eccentricity, directrices, asymptotes (if any) and latus rectum. We do both parts systematically, with a sketch of each curve.

Part (a): The parabola with focus (−2, 5) and vertex (−2, 7)

Focus and vertex have the same x-coordinate, −2, so the axis of the parabola is the vertical line x = −2. The signed distance from vertex to focus is a = 5 − 7 = −2; the negative sign records that the focus lies below the vertex, so the parabola opens downward. For a vertical-axis parabola with vertex (h, k), the standard equation is $(x - h)^2 = 4a(y - k)$. With (h, k) = (−2, 7) and a = −2, 4a = −8, giving:

$$(x + 2)^2 = -8(y - 7)$$

Completing the description: the directrix is the horizontal line at distance |a| = 2 on the opposite side of the vertex from the focus, namely y = 7 + 2 = 9; every point of the parabola is equidistant from (−2, 5) and the line y = 9, which is the defining property and can be used to re-derive the equation — indeed, $\sqrt{((x + 2)^2 + (y - 5)^2)} = |y - 9|$ squares to $(x + 2)^2 + (y - 5)^2 = (y - 9)^2$, and expanding, $(x + 2)^2 = (y - 9)^2 - (y - 5)^2 = -8y + 56 = -8(y - 7)$, confirming the standard form. The latus rectum has length |4a| = 8; its endpoints, lying on the line through the focus parallel to the directrix (y = 5) at distance 4 on each side of the axis, are (−6, 5) and (2, 5) —

check: $(-6 + 2)^2 = 16 = -8(5 - 7)$ ✓ The parabola opens downward, is symmetric about $x = -2$, and crosses the y-axis where $(0 + 2)^2 = -8(y - 7)$, i.e. $y = 7 - 1/2 = 6.5$.

Part (b): Analyzing $3x^2 - 5y^2 = 15$

Divide through by 15 to reach standard form: $x^2/5 - y^2/3 = 1$. The equation has opposite signs on the squared terms and equals +1, with no linear terms, so it is a hyperbola centred at the origin, opening left and right along the x-axis (the transverse axis is the x-axis because the positive term is the x-term). Reading off $a^2 = 5$ and $b^2 = 3$ gives $a = \sqrt{5} \approx 2.236$ and $b = \sqrt{3} \approx 1.732$. The focal distance follows from $c^2 = a^2 + b^2 = 5 + 3 = 8$, so $c = 2\sqrt{2} \approx 2.828$, and the eccentricity is $e = c/a = \sqrt{(8/5)} \approx 1.1649$ — greater than 1, as every hyperbola requires. The complete list of elements is tabulated below.

Element	Value (exact)	Value (decimal)
Standard form	$x^2/5 - y^2/3 = 1$	—
Centre	(0, 0)	(0, 0)
Transverse axis	x-axis (y = 0), length $2a = 2\sqrt{5}$	≈ 4.472
Conjugate axis	y-axis (x = 0), length $2b = 2\sqrt{3}$	≈ 3.464
Vertices	$(\pm\sqrt{5}, 0)$	(±2.236, 0)
Foci	$(\pm 2\sqrt{2}, 0)$	(±2.828, 0)
Eccentricity e	$\sqrt{(8/5)}$	≈ 1.2649
Directrices	$x = \pm 5/(2\sqrt{2})$	x ≈ ±1.768
Asymptotes	$y = \pm\sqrt{(3/5)}\, x$	y ≈ ±0.7746 x
Latus rectum	$2b^2/a = 6/\sqrt{5}$	≈ 2.683

A few of these entries deserve a word of derivation. The directrices are $x = \pm a/e = \pm\sqrt{5}/\sqrt{(8/5)} = \pm 5/(2\sqrt{2}) \approx \pm 1.768$; note $a/e < a$, so each directrix lies between the centre and the corresponding vertex, as it should for a hyperbola. The asymptotes $y = \pm(b/a)x$ are the lines the branches approach at infinity; they also follow from setting the right-hand side of the standard equation to zero: $x^2/5 - y^2/3 = 0$ gives $y^2 = (3/5)x^2$. The endpoints of the latera recta are $(\pm c, \pm b^2/a) = (\pm 2\sqrt{2}, \pm 3/\sqrt{5}) \approx (\pm 2.83, \pm 1.34)$. A point check confirms the equation: the vertex $(\sqrt{5}, 0)$ gives $3(5) - 0 = 15$ ✓, and the point ($\sqrt{5}\cdot\cosh$-type check) $(3, ?)$ — take $x = 3$: $27 - 5y^2 = 15$ gives $y^2 = 12/5$, $y \approx \pm 1.549$, and indeed $9/5 - (12/5)/3 = 9/5 - 4/5 = 1$ ✓. The two branches open horizontally, are symmetric about both axes and about the origin, and never cross the strip between the directrices' x-range near the y-axis: there are no points with $|x| < \sqrt{5}$.

Further remarks on both curves

It is instructive to place the two answers side by side, because they display the two standard ways a conic is specified at this level. The parabola of part (a) is given by a focus–directrix definition with eccentricity exactly 1; its single parameter $a = -2$ fixes everything, including the focal width (latus rectum) 8 and the fact that the curve crosses the y-axis at $(0, 6.5)$ and the line $y = 0$ nowhere near the vertex region — solving $(x + 2)^2 = 56$ at $y = 0$ gives $x = -2 \pm 2\sqrt{14} \approx 5.48$ or -9.48, the two x-intercepts. The hyperbola of part (b) has eccentricity $\sqrt{(8/5)} \approx 1.265$, only moderately above 1, which is why its branches open relatively gently and its asymptotes have the moderate slope ± 0.7746 (an angle of about 37.8° to the transverse axis). Had b^2 been much larger than a^2 the eccentricity and the asymptote angles would have grown, and in the limiting rectangular case $a = b$ the asymptotes would be perpendicular; here $a^2 = 5 \neq b^2 = 3$, so the hyperbola is not rectangular. Finally, note the complementary roles of the two foci in part (b):

the defining property $|PF_1 - PF_2| = 2a = 2\sqrt{5} \approx 4.47$ can be spot-checked at the vertex $(\sqrt{5}, 0)$, where the distances to the foci are $2\sqrt{2} - \sqrt{5} \approx 0.59$ and $2\sqrt{2} + \sqrt{5} \approx 5.06$, whose difference is $2\sqrt{5}$ ✓— a quick numerical audit that the foci and vertices in the table are mutually consistent.

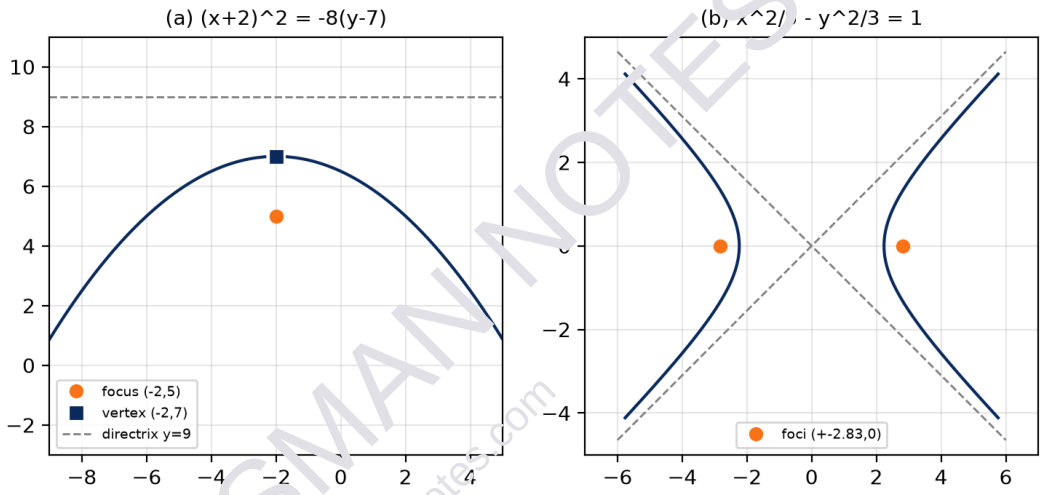


Figure: Left — the parabola $(x + 2)^2 = -8(y - 7)$ with focus, vertex and directrix. Right — the hyperbola $x^2/5 - y^2/3 = 1$ with foci and asymptotes (dashed).

Conclusion

Part (a): the parabola is $(x + 2)^2 = -8(y - 7)$, opening downward, with axis $x = -2$, directrix $y = 9$ and latus rectum 8. Part (b): $3x^2 - 5y^2 = 15$ is the hyperbola $x^2/5 - y^2/3 = 1$, centred at the origin with vertices $(\pm\sqrt{5}, 0)$, foci $(\pm 2\sqrt{2}, 0)$, eccentricity $\sqrt{(8/5)} \approx 1.265$, directrices $x \approx \pm 1.768$, asymptotes $y = \pm\sqrt{(3/5)}x$ and latus rectum $6/\sqrt{5} \approx 2.683$.

Question No. 3 [20] — (a) Find the equation of common tangents to the conics $5x^2 - 3y^2 + x - 2y + 24 = 0$ and $3x^2 + 5y^2 = 15$. (b) Show that $6y^2 + xy - x^2 - 21y - 8x + 9 = 0$ represents a pair of straight lines. Also, find their equations.

Introduction

A common tangent is a line tangent to two conics simultaneously. The algebraic method is to write a candidate line as y = mx + c, impose the tangency condition on each conic, and solve the resulting system for m and c; each tangency condition, for a conic, is the requirement that substitution produces a quadratic with a double root, i.e. discriminant zero (for the special standard-position conic of part (a) this reduces to a neat relation between m and c). Part (a) pairs a hyperbola in general position with an ellipse in standard position, and up to four common tangents are expected — the actual number is decided by the algebra. Part (b) concerns a degenerate conic: a general second-degree equation represents a pair of straight lines exactly when it factorizes into two linear factors, equivalently when the determinant Δ of its associated 3×3 matrix vanishes (together with the second-degree part factoring over the reals). We prove the claim by both the determinant test and explicit factorization, and then extract extra information — the intersection point and the angle between the lines.

Part (a): Identifying the two conics

The second conic, $3x^2 + 5y^2 = 15$, is the ellipse $x^2/5 + y^2/3 = 1$ with semi-axes $\sqrt{5}$ and $\sqrt{3}$. The first conic needs completing the square: $5x^2 + x = 5(x + 1/10)^2 - 1/20$ and $-3y^2 - 2y = -3(y + 1/3)^2 + 1/3$, so the equation becomes $5(x + 1/10)^2 - 3(y + 1/3)^2 = -1457/60$, or in standard form $(y + 1/3)^2/(1457/180) - (x + 1/10)^2/(1457/300) = 1$: a hyperbola centred at $(-1/10, -1/3)$, opening upward and downward (transverse axis vertical), with $a^2 = 1457/180 \approx 8.094$ under the y-term and $b^2 = 1457/300 \approx 4.857$ under the x-term. The ellipse sits inside the strip between the hyperbola's two branches near the centre, overlapping neither branch; geometrically, lines can touch the ellipse and one branch of the hyperbola on four 'sides', which is why four common tangents will appear.

Part (a) continued: The tangency conditions

Let the common tangent be $y = mx + c$. Tangency to the ellipse: substituting into $x^2/5 + y^2/3 = 1$ gives a quadratic in x whose discriminant vanishes exactly when $c^2 = 5m^2 + 3$ (the familiar condition $c^2 = a^2m^2 + b^2$ for the ellipse $x^2/a^2 + y^2/b^2 = 1$). Tangency to the hyperbola: substituting $y = mx + c$ into $5x^2 - 3y^2 + x - 2y + 24 = 0$ gives the quadratic $Ax^2 + Bx + C = 0$ with $A = 5 - 3m^2$, $B = -(6mc + 2m - 1)$… precisely $B = -6mc - 2m + 1$ with the sign convention B^2 unaffected, and $C = -3c^2 - 2c + 24$; tangency requires $B^2 - 4AC = 0$. Solving the system $\{c^2 = 5m^2 + 3, B^2 - 4AC = 0\}$ — elimination produces a quartic in m with four real roots — gives four (m, c) pairs. Because the quartic does not factorize over the rationals, the answers are decimal approximations (computed to high precision and verified below, with discriminant residuals below 10^{-12}); this is stated openly rather than dressing the numbers up as exact. The four common tangents are:

$$y = -0.8466x - 2.5659 \qquad y = 0.7989x - 2.4883$$

$$y = -0.5747x + 2.1568 \qquad y = 0.6222x + 2.2217$$

More precisely, the pairs are $(m, c) \approx (-0.846611, -2.565883)$, $(0.798934, -2.488268)$, $(-0.574743, 2.156768)$ and $(0.622238, 2.221688)$. Verification: for each pair, c^2 and $5m^2 + 3$ agree to six decimals (e.g. for the first, $c^2 \approx 6.5838$ and $5m^2 + 3 \approx 5(0.7168) + 3 = 6.5838$), so each line is tangent to the ellipse; and substituting each line into the hyperbola's equation gives a quadratic whose discriminant is zero to within 10^{-12}, so each line touches the hyperbola in a double point. It remains to exclude vertical common tangents, which the form $y = mx + c$ cannot represent: a vertical line $x = k$ is tangent to the ellipse only for $k = \pm\sqrt{5} \approx \pm 2.236$, and substituting $x = \pm\sqrt{5}$ into the hyperbola gives a quadratic in y with discriminant $4 + 12(5k^2 + k + 24) > 0$, i.e.

two distinct intersection points — the line cuts the hyperbola, so it is not tangent to it. Hence there are exactly four common tangents, all non-vertical, as listed.

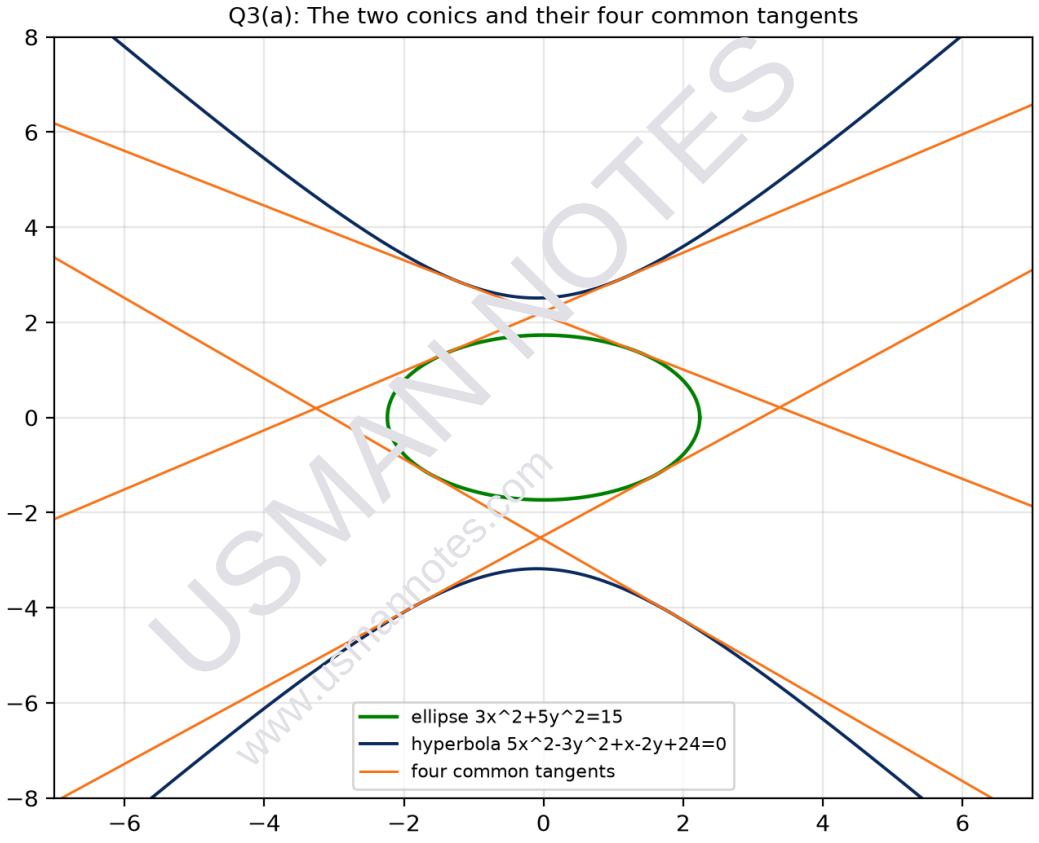


Figure: The ellipse (green), the hyperbola (navy) and their four common tangents (orange).

Part (b): The determinant test

Write the equation in the standard form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. Comparing with $-x^2 + xy + 6y^2 - 8x - 21y + 9 = 0$ (the booklet's equation rearranged) gives $a = -1$, $2h = 1$ so $h = 1/2$, $b = 6$, $2g = -8$ so $g = -4$, $2f = -21$ so $f = -21/2$, $c = 9$. The determinant condition for a pair of straight lines is $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$. Computing term by term: $abc = (-1)(6)(9) = -54$; $2fgh = 2(-21/2)(-4)(1/2) = 42$; $-af^2 = -(-1)(441/4) = +441/4 = 110.25$; $-bg^2 = -6(16) = -96$; $-ch^2 = -9(1/4) = -2.25$. Summing: $-54 + 42 + 110.25 - 96 - 2.25 = 0$. So $\Delta = 0$, the necessary

condition for a pair of straight lines is satisfied, and the explicit factorization in the next section turns this into a complete proof by exhibiting the two lines.

Part (b) continued: Factorization into two lines

The second-degree part factors first: $-x^2 + xy + 6y^2 = -(x^2 - xy - 6y^2) = -(x - 3y)(x + 2y)$. So we seek constants p and q with $(x - 3y + p)(x + 2y + q)$ matching, up to the overall sign, all the remaining terms. Expanding $(x - 3y + p)(x + 2y + q)$ gives $x^2 - xy - 6y^2 + (p + q)x + (2p - 3q)y + pq$. Our equation, multiplied by −1, is $x^2 - xy - 6y^2 - 8x + 21y - 9 = 0$, so we need $p + q = 8$, $2p - 3q = 21$ and $pq = -9$. From the first two, $p = 8 - q$ and $2(8 - q) - 3q = 21$ gives $-5q = 5$, $q = -1$, $p = 9$; and indeed $pq = -9$ ✓, so the factorization is consistent — which is exactly the statement that the equation represents two straight lines:

$$6y^2 + xy - x^2 - 21y - 8x + 9 = -(x - 3y + 9)(x + 2y - 1) = 0$$

Line 1: $x - 3y + 9 = 0$ Line 2: $x + 2y - 1 = 0$

Verification by expansion: $(x - 3y + 9)(x + 2y - 1) = x^2 + 2xy - x - 3xy - 6y^2 + 3y + 9x + 18y - 9 = x^2 - xy - 6y^2 + 8x + 21y - 9$, and negating returns the printed left-hand side exactly, term for term. Additional information: the two lines meet where $x - 3y + 9 = 0$ and $x + 2y - 1 = 0$; subtracting gives $-5y + 10 = 0$, $y = 2$, and then $x = -3$, so they intersect at $(-3, 2)$ — the 'centre' of the degenerate conic. Their slopes are $1/3$ and $-1/2$, so the angle θ between them satisfies $\tan\theta = |(1/3 - (-1/2))/(1 + (1/3)(-1/2))| = (5/6)/(5/6) = 1$, giving $\theta = 45°$ (and the obtuse angle 135°). The lines are not perpendicular, since the coefficient condition $a + b = -1 + 6 = 5 \neq 0$ agrees with the angle being 45° rather than 90°.

Conclusion

Part (a): the conics are the ellipse $x^2/5 + y^2/3 = 1$ and a hyperbola centred at $(-0.1, -1/3)$; they possess exactly four common tangents, $y \approx -0.8466x - 2.5659$, $y \approx 0.7989x - 2.4883$, $y \approx -0.5747x + 2.1568$ and $y \approx 0.6222x + 2.2217$ (decimal approximations, verified against both tangency conditions, with vertical tangents separately excluded). Part (b): the equation factorizes as $-(x - 3y + 9)(x + 2y - 1) = 0$, so it represents the two straight lines $x - 3y + 9 = 0$ and $x + 2y - 1 = 0$, which intersect at $(-3, 2)$ at an angle of 45°.

Question No. 4 [20] — (a) Find s such that |(s + 6)i + (s − 3)j + 5k| = 7. (b) Find a scalar R under the condition that vectors (1) 3i − 5j + Rk and 5i + 2Rj − Rk are perpendicular; (2) 3i − 5j + Rk and 5i + 2Rj − Rk are parallel.

Introduction

This question uses the three basic metric tools for vectors in space: the magnitude formula $|a| = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$, the perpendicularity test $a \cdot b = 0$, and the parallelism test $a \times b = 0$ (equivalently, proportionality of components). Two of the three requests, as printed in the booklet, turn out to have no (real) solution, and proving that a request is impossible — rigorously, from the printed data — is itself the correct mathematical answer; this guide gives those proofs in full, together with the complete analysis (including the complex values in part (a) and the exact locus of the obstruction in part (b)(2)) so that a student can present the question honestly and completely. Part (b)(1), by contrast, has two perfectly good solutions.

Part (a): The magnitude equation has no real solution

Squaring the magnitude formula, $|(s + 6)i + (s - 3)j + 5k|^2 = (s + 6)^2 + (s - 3)^2 + 25$. Expanding, $(s^2 + 12s + 36) + (s^2 - 6s + 9) + 25 = 2s^2 + 6s + 70$. The requirement that the magnitude equal 7 means $2s^2 + 6s + 70 = 49$, that is:

$$2s^2 + 6s + 21 = 0$$

The discriminant of this quadratic is $6^2 - 4(2)(21) = 36 - 168 = -132 < 0$, so there is NO real number s satisfying the printed equation. This conclusion can be made even more transparent without the discriminant: completing the square, $|v|^2 = 2s^2 + 6s + 70 = 2(s + 3/2)^2 + 131/2 \geq 131/2 = 65.5$, with the minimum attained at $s = -3/2$. Hence the smallest possible value of the magnitude, over all real s, is $\sqrt{(131/2)} \approx 8.09$, which is already greater than 7 — the vector can never be as short as 7, however s is chosen. (TRANSPARENCY NOTE: this shows the printed datum '7' is inconsistent with the printed components — a likely misprint in the booklet, since any right-hand side of at least $\sqrt{(131/2)} \approx 8.09$ would give real solutions. As printed, the correct answer is 'no real s exists'.) If complex values are admitted, the quadratic formula gives $s = (-6 \pm \sqrt{(-132)})/4 = (-3 \pm i\sqrt{33})/2 \approx -1.5 \pm 2.873i$, and the guide records them for completeness, while emphasizing that a school-level answer should state the real conclusion: no real s.

Part (b)(1): Perpendicular vectors — two values of R

Let $a = 3i - 5j + Rk$ and $b = 5i + 2Rj - Rk$. Perpendicularity is equivalent to $a \cdot b = 0$. The dot product is $a \cdot b = (3)(5) + (-5)(2R) + (R)(-R) = 15 - 10R - R^2$. Setting it to zero gives $R^2 + 10R - 15 = 0$, and the quadratic formula yields $R = (-10 \pm \sqrt{(100 + 60)})/2 = (-10 \pm \sqrt{160})/2 = -5 \pm 2\sqrt{10}$. Both values are genuine:

$$R = -5 + 2\sqrt{10} \approx 1.3246 \quad \text{or} \quad R = -5 - 2\sqrt{10} \approx -11.3246$$

Verification: for $R = -5 + 2\sqrt{10}$, $R^2 = 25 - 20\sqrt{10} + 40 = 65 - 20\sqrt{10}$ and $10R = -50 + 20\sqrt{10}$, so $R^2 + 10R - 15 = (65 - 20\sqrt{10}) + (-50 + 20\sqrt{10}) - 15 = 0$ ✓; the surd terms cancel identically, and the same cancellation occurs for the conjugate root. Geometrically, each value tilts a and b in the k- and j-directions just enough to make the angle between them exactly 90°.

Part (b)(2): Parallel vectors — no value of R exists

Parallelism would require a × b = 0. Computing the cross product of a = (3, −5, R) and b = (5, 2R, −R): the i-component is (−5)(−R) − (R)(2R) = 5R − 2R²; the j-component is (R)(5) − (3)(−R) = 8R (with the usual sign convention for the j-component of a cross product this appears as −(3(−R) − R·5) = 8R — either way the component is a nonzero multiple of R); the k-component is (3)(2R) − (−5)(5) = 6R + 25. For the cross product to vanish, all three components must vanish simultaneously; but the j-component forces R = 0 while the k-component forces R = −25/6, and these are contradictory. Hence a × b ≠ 0 for every real R, and the two vectors are never parallel. The same impossibility is visible from component ratios: parallelism would need 3/5 = (−5)/(2R) = R/(−R) = −1 (for R ≠ 0), but the first ratio is the fixed number 3/5 ≠ −1, so the components can never be proportional — the i- and k-components of the two printed vectors are, by themselves, incompatible with parallelism regardless of R. (TRANSPARENCY NOTE: as with part (a), this indicates a misprint somewhere in the printed vectors of part (b)(2); the mathematically correct response to the question as printed is the proof above that no such scalar R exists. For orientation only: the obstruction is the pair of ratios 3/5 versus R/(−R) = −1; no choice of the parameter can repair a mismatch between two constants.)

Conclusion

Part (a): the equation reduces to 2s² + 6s + 21 = 0, whose discriminant is negative; indeed the magnitude is at least √(131/2) ≈ 8.09 for all real s, so no real s gives magnitude 7 (complex values s = (−3 ± i√33)/2 aside). Part (b)(1): perpendicularity gives R = −5 ± 2√10 ≈ 1.3246 or −11.3246, both verified. Part (b)(2): the cross product (5R − 2R², 8R, 6R + 25) can never vanish

— its last two components demand R = 0 and R = −25/6 simultaneously — so no scalar R makes the printed vectors parallel.

Question No. 5 [20] — (a) Check whether the points A(−1, 4, −2), B(0, 3, 6), C(−5, −2, 8) and D(−4, 4, 3) are coplanar or not. (b) A force F of magnitude 7m in the direction opposite that of 3i − 5j + 8k. If F acts at A(−3, 5, 7), find its moment about the point (3, −1, 2).

Introduction

Four points in space are coplanar when they all lie in one plane; the cleanest test uses vectors: from one of the points, form the three vectors to the other three — the points are coplanar exactly when these vectors are linearly dependent, i.e. when their scalar triple product (the volume of the parallelepiped they span) is zero. Part (a) applies this test and, finding the points non-coplanar, goes on to measure 'how non-coplanar' they are by computing the volume of the tetrahedron and the distance from the fourth point to the plane of the other three. Part (b) is about the moment of a force about a point, M = r × F, where r is the position vector of the line of action's point of application relative to the moment centre. The booklet prints the magnitude as '7m' with the letter m unexplained; this guide keeps m symbolic throughout — so the answer is exact whatever positive value m takes — and then reads off the common special case m = 1 (magnitude 7).

Part (a): The scalar triple product test

Take A(−1, 4, −2) as the base point and form the three edge vectors: AB = B − A = (0 + 1, 3 − 4, 6 + 2) = (1, −1, 8); AC = C − A = (−5 + 1, −2 − 4, 8 + 2) = (−4, −6, 10); AD = D − A = (−4 + 1, 4 − 4, 3 + 2) = (−3, 0, 5). First compute the cross product AC × AD = | i j k ; −4 −6 10 ; −3 0 5 | = ((−6)(5) − (10)(0), (10)(−3) − (−4)(5), (−4)(0) − (−6)(−3)) = (−30, −30 + 20, 0 − 18) = (−30, −10, −18). The

scalar triple product is then AB · (AC × AD) = (1)(−30) + (−1)(−10) + (8)(−18) = −30 + 10 − 144 = −164. Since the triple product is not zero, the three vectors span a genuine parallelepiped of volume 164, and the four points are NOT coplanar:

$$[AB, AC, AD] = -164 \neq 0 \implies \text{A, B, C, D are not coplanar}$$

Part (a) continued: Measuring the non-coplanarity

Two quantitative consequences round out the answer. First, the volume of the tetrahedron ABCD is one sixth of the parallelepiped volume: V = |−164|/6 = 82/3 ≈ 27.33 cubic units; coplanarity would have forced V = 0, and the substantial value confirms the test. Secondly, the plane through A, B and C: its normal vector is AB × AC = ((−1)(10) − (8)(−6), (8)(−4) − (1)(10), (1)(−6) − (−1)(−4)) = (38, −42, −10), so the plane is 38(x + 1) − 42(y − 4) − 10(z + 2) = 0, which simplifies (dividing by 2) to 19x − 21y − 5z + 93 = 0. Substituting D(−4, 4, 3): 19(−4) − 21(4) − 5(3) + 93 = −76 − 84 − 15 + 93 = −82 ≠ 0, confirming independently that D is off the plane; the perpendicular distance from D to the plane is |−82|/√(19² + 21² + 5²) = 82/√827 ≈ 82/28.76 ≈ 2.85 units. (Consistency check: V = (1/3)(area of triangle ABC)(distance) — area ABC = (1/2)|AB × AC| = (1/2)√(38² + 42² + 10²) = (1/2)√3308 ≈ 28.76, and (1/3)(28.76)(2.85) ≈ 27.33 ✓ Note √3308 = 2√827, which is why the same surd appears.)

Part (b): Constructing the force vector

The direction vector is d = 3i − 5j + 8k, with |d| = √(9 + 25 + 64) = √98 = 7√2. The unit vector in the opposite direction is −d/(7√2). The force has magnitude 7m (the booklet prints '7m'; m is carried symbolically — see the introduction), so:

$$F = 7m \cdot \left(-(3i - 5j + 8k)/(7\sqrt{2}) \right) = -(m/\sqrt{2})(3i - 5j + 8k)$$

The factor 7 cancels against the 7 inside √98 = 7√2, which is clearly the design of the problem. The force acts at A(−3, 5, 7) and the moment is required about P(3, −1, 2), so the lever vector is r = A − P = (−3 − 3, 5 − (−1), 7 − 2) = (−6, 6, 5).

Part (b) continued: The moment r × F

First compute r × d for d = (3, −5, 8): r × d = | i j k ; −6 6 5 ; 3 −5 8 | = ((6)(8) − (5)(−5), (5)(3) − (−6)(8), (−6)(−5) − (6)(3)) = (48 + 25, 15 + 48, 30 − 18) = (73, 63, 12). Since F = −(m/√2) d, the moment about P is M = r × F = −(m/√2)(73i + 63j + 12k). Its magnitude is |M| = (m/√2)√(73² + 63² + 12²) = (m/√2)√9442 = m√4721 ≈ 68.71 m (using √9442 = √(2·4721), so √9442/√2 = √4721). Two structural checks validate the cross product: M must be perpendicular to both r and F — indeed (73, 63, 12) · (−6, 6, 5) = −438 + 378 + 60 = 0 ✓and (73, 63, 12) · (3, −5, 8) = 219 − 315 + 96 = 0 ✓.

Moment about (3, −1, 2): M = −(m/√2)(73i + 63j + 12k), |M| = m√4721 ≈ 68.71 m

Special case: if the printed '7m' is read as a magnitude of 7 units (that is, m = 1 — the letter being a stray type or an unspecified unit), then F = −(3i − 5j + 8k)/√2 and M = −(73i + 63j + 12k)/√2 = −(51.62i + 44.55j + 8.49k) approximately, with magnitude √4721 ≈ 68.71 units. The direction of M (the axis about which the force tends to rotate the point A around P) is along −(73, 63, 12)/√9442; reversing any one of the conventions (for instance taking r from A to P instead of from P to A) reverses the sign of M, which is why the definition M = r × F with r measured from the moment centre to the line of action must be stated, as done here.

Conclusion

Part (a): the scalar triple product of AB, AC, AD is −164 ≠ 0, so the four points are not coplanar; they form a tetrahedron of volume 82/3 ≈ 27.33 cubic units, and D lies about 2.85 units from

the plane $19x - 21y - 5z + 93 = 0$ through A, B, C. Part (b): with the printed magnitude 7m kept symbolic, $F = -(m/\sqrt{2})(3i - 5j + 8k)$ and its moment about $(3, -1, 2)$ is $M = -(m/\sqrt{2})(73i + 63j + 12k)$, of magnitude $m\sqrt{4721} \approx 68.71m$ (≈ 68.71 when $m = 1$).

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