ALLAMA IQBAL OPEN UNIVERSITY

USMAN NOTES
www.usmannotes.com

Name:

Registration No.:

Course Code: 1309

Program: Intermediate (HSSC)

Semester: Autumn 2026

ALLAMA IQBAL OPEN UNIVERSITY, ISLAMABAD

Course: Mathematics-III (1309)

Programme / Level: Intermediate (HSSC)

Department of Mathematics

Semester: Autumn 2026

Assignment No. 1 (Units 1–5)

Total Marks: 100 Pass Marks: 40

SOLVED ASSIGNMENT GUIDE — For Study & Reference

Note for students: This guide explains the method and the reasoning behind every step so that you can learn the topic properly. Please study the working, then write your assignment in your own words and in your own hand; copying any guide word-for-word is against the University's rules.

Question No. 1 [20] — (a) Find δy and dy if y = tan √x and when x changes from π/6 to π/3. (b) Integrate ∫ 2x sin x² cot x² dx.

Introduction

This question tests two closely related ideas from differential calculus: the actual increment of a function and its differential. When the independent variable x changes by a finite amount δx, the function y = f(x) changes by the exact amount δy = f(x + δx) − f(x); this is the true, measurable change along the curve. The differential dy, by contrast, is the change predicted by the tangent line at the starting point: dy = f ′(x) δx. The differential is a linear approximation of the increment, and the two agree closely only when δx is small. Part (a) asks us to compute both quantities exactly for the function y = tan √x while x moves from π/6 to π/3, a change that

is not particularly small, so we should expect a visible gap between δy and dy, and part of the educational value of the question is precisely to measure that gap. Part (b) is an integration problem whose integrand collapses, after one trigonometric simplification, into the derivative of a simple composition, so it is solved by a single substitution.

Part (a): The increment δx and the two values of the function

The variable x changes from $x_1 = \pi/6$ to $x_2 = \pi/3$, so the increment of x is $\delta x = x_2 - x_1 = \pi/3 - \pi/6 = \pi/6 \approx 0.523599$. To find the exact change in y we evaluate the function at the two endpoints. The square roots come first: $\sqrt{(\pi/6)} = \sqrt{0.523599} \approx 0.723601$ radians, and $\sqrt{(\pi/3)} = \sqrt{1.047198} \approx 1.023328$ radians. Note carefully that the tangent is being taken of the square root of x, not of x itself; the argument of the tangent function is √x, measured in radians, because all calculus formulas for trigonometric derivatives assume radian measure.

$$y_1 = \tan \sqrt{(\pi/6)} = \tan (0.723601) \approx 0.883460$$

$$y_2 = \tan \sqrt{(\pi/3)} = \tan (1.023328) \approx 1.640342$$

The actual increment of the function is therefore the difference of these two values:

$$\delta y = y_2 - y_1 \approx 1.640342 - 0.883460 = 0.756882$$

This is the exact change: as x increases by about 0.5236, the function y = tan √x increases by about 0.7569. No approximation has been made in δy beyond the rounding of the final decimals; it is the genuine vertical distance travelled along the curve between the two points.

Part (a) continued: The derivative and the differential dy

The differential uses the derivative at the starting point $x = \pi/6$. The function is a composition, y = tan(u) with u = √x = x^(1/2), so the chain rule gives $dy/dx = \sec^2(\sqrt{x}) \cdot (1/(2\sqrt{x}))$, because the derivative of tan u is $\sec^2 u$ and the derivative of √x is $1/(2\sqrt{x})$. Evaluating at $x = \pi/6$, we need

$\sec^2(0.723601)$; using the identity $\sec^2\theta = 1 + \tan^2\theta$ and the value $\tan(0.723601) \approx 0.883460$ found above, $\sec^2(\sqrt{(\pi/6)}) \approx 1 + (0.883460)^2 \approx 1 + 0.780501 = 1.780501$. Dividing by $2\sqrt{(\pi/6)} \approx 1.447202$ gives the slope of the tangent line at the starting point:

$$dy/dx \text{ at } x = \pi/6 = \sec^2(\sqrt{x}) / (2\sqrt{x}) \approx 1.780501 / 1.447202 \approx 1.230310$$

The differential is this slope multiplied by the increment $\delta x = \pi/6 \approx 0.523599$:

$$dy = f'(\pi/6) \cdot \delta x \approx 1.230310 \times 0.523599 \approx 0.644186$$

Comparison of δy and dy — discussion

We have $\delta y \approx 0.756882$ (exact change) against $dy \approx 0.644186$ (tangent-line prediction). The differential underestimates the true change by about $0.756882 - 0.644186 = 0.112696$, which is roughly 14.9 percent of δy. The reason is geometric: the curve $y = \tan\sqrt{x}$ is bending upward (its slope keeps increasing, because both $\sec^2(\sqrt{x})$ grows and the factor $1/(2\sqrt{x})$ changes slowly while the dominant effect is the growth of $\sec^2$), so the tangent line at the left-hand point lies below the curve, and travelling along the tangent for a fairly large step $\delta x = \pi/6$ lands below the true endpoint. Had δx been tiny — say 0.001 — the two quantities would have agreed to many decimal places. This is the standard lesson of differentials: dy is an excellent local approximation of δy, but it is not identical to δy for finite steps, and the error grows with the step size and with the curvature of the graph. In examinations students often write $\delta y = dy$ as if they were the same; this numerical example shows concretely why that is wrong.

Part (b): Integrating $\int 2x \sin x^2 \cot x^2 \, dx$

Here $\sin x^2$ and $\cot x^2$ mean $\sin(x^2)$ and $\cot(x^2)$: the trigonometric functions act on the quantity x^2. The key simplification comes from the definition of the cotangent as the ratio of cosine to

sine: $\cot(x^2) = \cos(x^2)/\sin(x^2)$. Multiplying by $\sin(x^2)$ cancels the denominator exactly, so the whole integrand simplifies before any integration is attempted:

$$\sin x^2 \cdot \cot x^2 = \sin(x^2) \cdot \cos(x^2)/\sin(x^2) = \cos(x^2)$$

$$\int 2x \sin x^2 \cot x^2 \, dx = \int 2x \cos(x^2) \, dx$$

Now the factor $2x$ is precisely the derivative of the inner function x^2, which is the fingerprint of the substitution method. Put $u = x^2$, so $du = 2x\,dx$ and the integral becomes the elementary $\int \cos u \, du = \sin u + C$. Returning to x gives the answer.

$$\int 2x \sin x^2 \cot x^2 \, dx = \sin(x^2) + C$$

Verification by differentiation: $d/dx\,[\sin(x^2)] = \cos(x^2) \cdot 2x$, by the chain rule, and $2x \cos(x^2) = 2x \sin(x^2) \cot(x^2)$ by the same cotangent identity run backwards, so the original integrand is recovered and the integration is confirmed. (Strictly, the identity $\sin\theta \cot\theta = \cos\theta$ needs $\sin(x^2) \neq 0$, so the formula holds on any interval where x^2 is not a multiple of π; on such an interval the constant C may be chosen independently.)

Conclusion

For part (a), the exact increment is $\delta y \approx 0.7569$ while the differential is $dy \approx 0.6442$; the differential underestimates the true change by about 0.1127 because the increment $\delta x = \pi/6$ is large and the curve bends away from its tangent. For part (b), one trigonometric identity reduces the integrand to $2x \cos(x^2)$, and one substitution gives $\int 2x \sin x^2 \cot x^2 \, dx = \sin(x^2) + C$, verified by differentiating the answer.

Question No. 2 [20] — (a) Integrate $\int e^x (1 - \tan^2 x)/(1 + \tan^2 x)\,dx$. (b) Evaluate $\int_0^{(\pi/3)} (\cos x + 1)/\sin^2 x \,dx$.

Introduction

Part (a) is an indefinite integral in which a trigonometric fraction must first be recognised as a double-angle function; once that is done, the remaining integral of $e^x \cos 2x$ is a standard 'cyclic' integration-by-parts problem whose answer can also be written down from the general formula for ∫ e^(ax) cos(bx) dx. Part (b) looks like a routine definite integral, but it contains a trap that this guide is obliged to point out honestly: as printed in the booklet, with lower limit 0, the integral does not converge, because the integrand blows up like $2/x^2$ near $x = 0$. We therefore do two things in part (b): first, we compute the antiderivative correctly and use it to prove rigorously that the printed integral diverges to $+\infty$; secondly, as a clearly labelled remark, we record the finite formula that the same antiderivative gives on any interval $[a, \pi/3]$ with $a > 0$, which is what a student needs if the lower limit was misprinted.

Part (a): Simplifying the trigonometric fraction

Divide numerator and denominator of $(1 - \tan^2 x)/(1 + \tan^2 x)$ by $\cos^2 x$. Since $\tan^2 x = \sin^2 x/\cos^2 x$, the numerator becomes $(\cos^2 x - \sin^2 x)/\cos^2 x$ and the denominator becomes $(\cos^2 x + \sin^2 x)/\cos^2 x = 1/\cos^2 x$. The $\cos^2 x$ factors cancel, leaving $\cos^2 x - \sin^2 x$, which is the double-angle identity $\cos 2x$. Hence the integrand is simply $e^x \cos 2x$:

$$(1 - \tan^2 x)/(1 + \tan^2 x) = \cos^2 x - \sin^2 x = \cos 2x$$

$$I = \int e^x \cos 2x \, dx$$

Part (a) continued: Integration by parts, twice

Integrate by parts with $u = e^x$ and $dv = \cos 2x \, dx$, so $v = (1/2) \sin 2x$. This gives $I = (1/2) e^x \sin 2x - (1/2) \int e^x \sin 2x \, dx$. Apply parts once more to the remaining integral, again taking $u = e^x$: $\int e^x \sin 2x \, dx = -(1/2) e^x \cos 2x + (1/2) \int e^x \cos 2x \, dx = -(1/2) e^x \cos 2x + (1/2) I$. Substituting back, $I = (1/2) e^x \sin 2x - (1/2)[-(1/2) e^x \cos 2x + (1/2) I] = (1/2) e^x \sin 2x + (1/4) e^x \cos 2x - (1/4) I$. The

unknown integral I has reappeared on the right — this is the 'cycle' typical of exponential-times-trigonometric integrals — so we collect it: I + (1/4) I = (5/4) I, and dividing by 5/4 gives the result:

$$\int e^x (1 - \tan^2 x)/(1 + \tan^2 x)\, dx = e^x (\cos 2x + 2 \sin 2x)/5 + C$$

The same answer follows at once from the standard formula $\int e^{ax} \cos(bx)\, dx = e^{ax}(a \cos bx + b \sin bx)/(a^2 + b^2)$ with a = 1, b = 2, which yields the denominator 1 + 4 = 5. Verification by differentiation: differentiating $e^x(\cos 2x + 2 \sin 2x)/5$ gives $(1/5)[\, e^x(\cos 2x + 2 \sin 2x) + e^x(-2 \sin 2x + 4 \cos 2x)\,] = (1/5)\, e^x (5 \cos 2x) = e^x \cos 2x$, and since cos 2x equals the printed fraction $(1 - \tan^2 x)/(1 + \tan^2 x)$, the integration is confirmed exactly.

Part (b): The antiderivative

Split the integrand into two standard pieces: $(\cos x + 1)/\sin^2 x = \cos x/\sin^2 x + 1/\sin^2 x = \cos x \cdot \csc^2 x$… more precisely, the first piece is $\cos x/\sin^2 x$ and the second is $\csc^2 x$. For the first piece, substitute u = sin x, du = cos x dx, giving $\int du/u^2 = -1/u = -1/\sin x = -\csc x$. The second piece is the standard integral $\int \csc^2 x\, dx = -\cot x$. Hence an antiderivative on any interval not containing a multiple of π is:

$$F(x) = \int (\cos x + 1)/\sin^2 x\, dx = -(\csc x + \cot x) = -(1 + \cos x)/\sin x$$

A useful equivalent form, obtained from the half-angle identities $1 + \cos x = 2 \cos^2(x/2)$ and $\sin x = 2 \sin(x/2) \cos(x/2)$, is $F(x) = -\cot(x/2)$. Differentiating $-\cot(x/2)$ returns $(1/2) \csc^2(x/2) = 1/(1 - \cos x)$… the reader may check instead that $d/dx\, [-(1 + \cos x)/\sin x]$ simplifies, by the quotient rule, to $(\sin^2 x + (1 + \cos x) \cos x)/\sin^2 x = (1 + \cos x)/\sin^2 x$, since $\sin^2 x + \cos^2 x = 1$; so F is correct.

Part (b) continued: The printed integral diverges — an honest analysis

A NOTE ON THE PRINTED QUESTION (transparency): the lower limit printed in the booklet is 0, and at x = 0 the integrand is undefined, because sin 0 = 0 while the numerator cos 0 + 1 = 2 ≠ 0. The integral must therefore be treated as an improper integral, $\int_0^{\pi/3} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^{\pi/3}$. Using the antiderivative, $\int_\varepsilon^{\pi/3} = F(\pi/3) - F(\varepsilon) = -\sqrt{3} + (1 + \cos\varepsilon)/\sin\varepsilon = \cot(\varepsilon/2) - \sqrt{3}$, because $F(\pi/3) = -(1 + 1/2)/(\sqrt{3}/2) = -\sqrt{3}$. But as $\varepsilon \to 0^+$, $\cot(\varepsilon/2) \to +\infty$ (indeed $(1 + \cos\varepsilon)/\sin\varepsilon$ behaves like $2/\varepsilon$), so the integral grows beyond every bound: it diverges to $+\infty$ and has no finite value. A comparison argument confirms this: on $(0, \pi/3]$ the numerator satisfies $\cos x + 1 \geq 3/2$ and $\sin x \leq x$, so the integrand is at least $(3/2)/x^2$, and $\int_0 x^{-2}\,dx$ is the classic divergent integral. Any answer key that reports a finite number for the integral exactly as printed has silently ignored the singularity at the lower limit.

For completeness, the correct finite statement that the same working provides is: for any lower limit a with $0 < a \leq \pi/3$, $\int_a^{\pi/3} (\cos x + 1)/\sin^2 x\,dx = \cot(a/2) - \sqrt{3}$. For example — and this is offered only as an illustration of the formula, not as the printed question — if the intended lower limit had been $\pi/6$, the value would be $\cot(\pi/12) - \sqrt{3} = (2 + \sqrt{3}) - \sqrt{3} = 2$ exactly. Students meeting this question in the examination hall should present the antiderivative, note the divergence at 0 as shown above, and state the general formula; that is the mathematically correct response to the question as printed.

Conclusion

Part (a) simplifies, via the double-angle identity, to $\int e^x \cos 2x\,dx = e^x(\cos 2x + 2\sin 2x)/5 + C$, confirmed by differentiation. Part (b) has antiderivative $-(\csc x + \cot x) = -\cot(x/2)$; because the integrand is unbounded like $2/x^2$ near the printed lower limit 0, the definite integral as printed diverges to $+\infty$, while on any interval $[a, \pi/3]$ with $a > 0$ its value is $\cot(a/2) - \sqrt{3}$.

Question No. 3 [20] — (a) Find the point P(x, y) on the joint of A(3, −7) and B(−4, 5) that is twice as far from A as B is from A and lies on the same side of A as B does. (b) Find the point C(x, y) three-fifth of the way along the line segment from A(−2, 5) to B(4, 7). Hence, find the equation of the line through C(x, y) and perpendicular to the line 7x − 5y + 13 = 0.

Introduction

Both parts of this question are applications of one idea: travelling along the straight line through two points in proportion to distance. If we write a point on the line through A and B as A + t(B − A), then the parameter t measures exactly the fraction of the journey from A to B: t = 0 gives A, t = 1 gives B, t = 1/2 gives the midpoint, values between 0 and 1 give interior points of the segment, and values greater than 1 give points beyond B on the same side. Part (a) uses t = 2, part (b) uses t = 3/5, and part (b) then continues with a second standard idea — the equation of a line through a known point with a slope perpendicular to a given line. Keeping the parameter picture in mind makes both computations transparent and provides built-in checks.

Part (a): Twice as far from A, on the same side as B

The displacement from A(3, −7) to B(−4, 5) is B − A = (−4 − 3, 5 − (−7)) = (−7, 12), and the distance AB = √((−7)² + 12²) = √(49 + 144) = √193 ≈ 13.89. The required point P lies on the line through A and B, on the same side of A as B — that is, in the direction of the displacement (−7, 12), not the opposite direction — and at twice the distance: AP = 2·AB. In parameter form, P = A + 2(B − A), because scaling the displacement by 2 scales the distance by 2 while keeping the direction (and hence the side of A) unchanged:

$$P = (3, -7) + 2(-7, 12) = (3 - 14, -7 + 24) = (-11, 17)$$

Verification: AP = √((−11 − 3)² + (17 + 7)²) = √((−14)² + 24²) = √(196 + 576) = √772 = 2√193 = 2·AB, exactly as required. There is also an elegant structural check: the midpoint of A and P is ((3 − 11)/2, (−7 + 17)/2) = (−4, 5) = B, so B is the midpoint of AP — which is exactly what 'P is twice as far from A as B is, on the same side' must mean. In the language of the section formula, P divides AB externally in the ratio AP : PB = 2 : 1 (P is outside the segment, beyond B), and the external-division formula gives the same coordinates: ((2(−4) − 1(3))/(2 − 1), (2(5) − 1(−7))/(2 − 1)) = (−11, 17). The phrase 'on the same side of A as B does' is essential: without it, the point A − 2(B − A) = (17, −31) would be equally far from A but on the opposite side, and the question deliberately excludes it.

Part (b): Three-fifth of the way from A(−2, 5) to B(4, 7)

The displacement is B − A = (4 − (−2), 7 − 5) = (6, 2). Three-fifth of the way means the parameter t = 3/5 applied to this displacement, added to the starting point A:

$$C = A + (3/5)(B - A) = (-2, 5) + (3/5)(6, 2) = (-2 + 18/5, 5 + 6/5) = (8/5, 31/5) = (1.6, 6.2)$$

Verification by the section formula: C divides AB internally in the ratio AC : CB = 3 : 2, so C = ((3·4 + 2·(−2))/(3 + 2), (3·7 + 2·5)/(3 + 2)) = ((12 − 4)/5, (21 + 10)/5) = (8/5, 31/5), in agreement. A distance check confirms the fraction: AC = √((18/5)² + (6/5)²) = √(324 + 36)/5 = √360/5 = (6√10)/5, while AB = √(36 + 4) = √40 = 2√10, and AC/AB = (6√10/5)/(2√10) = 3/5, exactly three-fifth of the way, as required.

Part (b) continued: The line through C perpendicular to 7x − 5y + 13 = 0

The given line 7x − 5y + 13 = 0 can be written as y = (7/5)x + 13/5, so its slope is $m_1 = 7/5$. A perpendicular line has slope m_2 with $m_1 m_2 = -1$, hence $m_2 = -5/7$. Equivalently — and this is the quickest reliable method — a line perpendicular to ax + by + c = 0 has the form bx − ay + k = 0

with the coefficients swapped and one sign changed; here that gives 5x + 7y + k = 0 (check: its slope is −5/7, as required). It remains to fix k by demanding that the line pass through C(8/5, 31/5): 5(8/5) + 7(31/5) + k = 0 gives 8 + 217/5 + k = 0, so k = −257/5. Multiplying through by 5 to clear fractions:

$$25x + 35y - 257 = 0$$

Verification: substituting C, 25(8/5) + 35(31/5) − 257 = 40 + 217 − 257 = 0, so C lies on the line; and the product of the slopes, (7/5)(−5/7) = −1, confirms perpendicularity to the given line. Either check alone would catch most slips, and together they make the answer certain.

Conclusion

Part (a): the point is P(−11, 17), with B the midpoint of AP, so AP = 2 AB on the same side of A as B. Part (b): the point three-fifth of the way is C(8/5, 31/5) = (1.6, 6.2), and the line through C perpendicular to 7x − 5y + 13 = 0 is 25x + 35y − 257 = 0. Both parts are the same parameter idea — A + t(B − A) with t = 2 and t = 3/5 respectively — followed in part (b) by the perpendicular-slope condition.

Question No. 4 [20] — (a) Find circumcenter of the triangle with vertices A(7, −2), B(−2, 1) and C(3, 4). (b) Measure interior angles of the triangle with vertices A(1, −2), B(2, 3) and C(6, 5). Also, find the area of this triangle.

Introduction

The circumcenter of a triangle is the centre of the unique circle passing through all three vertices; equivalently, it is the point equidistant from the three vertices, and it is found as the intersection of the perpendicular bisectors of any two sides. Part (a) applies this method and then verifies the answer by computing all three distances. Part (b) concerns a different triangle

and asks for its three interior angles and its area: the standard route is to compute the three side lengths with the distance formula, obtain the angles from the cosine rule, and obtain the area from the coordinate (shoelace) formula, with the angle sum of 180° and the formula (1/2)·(product of two sides)·(sine of the included angle) serving as independent cross-checks. Both parts reward systematic verification, which we carry out in full.

Part (a): Perpendicular bisectors and the circumcenter

Side AB joins A(7, −2) and B(−2, 1); its midpoint is ((7 − 2)/2, (−2 + 1)/2) = (5/2, −1/2) and its slope is (1 − (−2))/(−2 − 7) = 3/(−9) = −1/3, so the perpendicular bisector of AB has slope 3 and equation y + 1/2 = 3(x − 5/2), that is, y = 3x − 8. Side AC joins A(7, −2) and C(3, 4); its midpoint is (5, 1) and its slope is (4 − (−2))/(3 − 7) = 6/(−4) = −3/2, so the perpendicular bisector of AC has slope 2/3 and equation y − 1 = (2/3)(x − 5). The circumcenter O lies on both bisectors, so substitute y = 3x − 8 into the second: 3x − 8 − 1 = (2/3)(x − 5), giving 3x − 9 = (2x − 10)/3, hence 9x − 27 = 2x − 10, so 7x = 17 and x = 17/7. Then y = 3(17/7) − 8 = 51/7 − 56/7 = −5/7. Therefore:

$$\text{Circumcenter } O = (17/7, -5/7) \approx (2.4286, -0.7143)$$

Verification by distances. $OA^2 = (7 - 17/7)^2 + (-2 + 5/7)^2 = (32/7)^2 + (-9/7)^2 = (1024 + 81)/49 = 1105/49$. $OB^2 = (-2 - 17/7)^2 + (1 + 5/7)^2 = (-31/7)^2 + (12/7)^2 = (961 + 144)/49 = 1105/49$. $OC^2 = (3 - 17/7)^2 + (4 + 5/7)^2 = (4/7)^2 + (33/7)^2 = (16 + 1089)/49 = 1105/49$. All three are equal, as they must be, and the circumradius is $R = \sqrt{1105}/7 \approx 4.7488$. The circumcircle itself is $(x - 17/7)^2 + (y + 5/7)^2 = 1105/49$. The figure below shows the triangle, its circumcenter and the circumcircle.

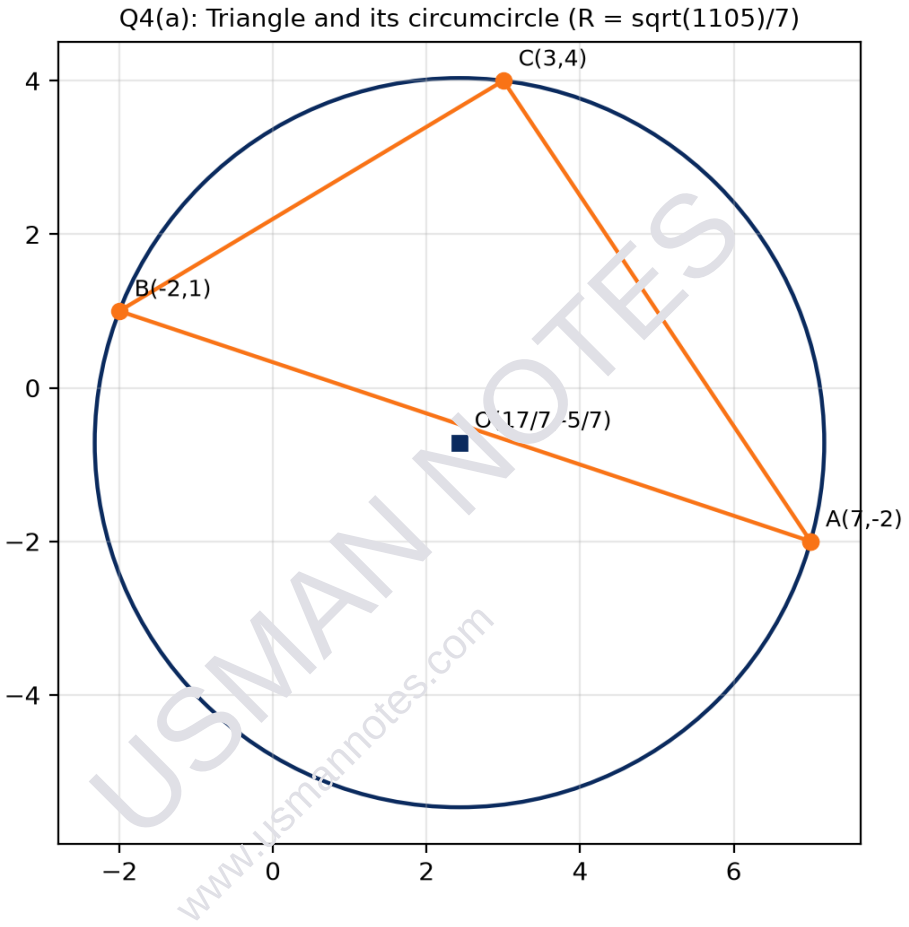


Figure: The triangle A(7, −2), B(−2, 1), C(3, 4) with circumcenter O(17/7, −5/7) and its circumcircle.

Part (b): Side lengths of the second triangle

For the triangle with vertices A(1, −2), B(2, 3), C(6, 5), denote the side opposite A by a = BC, opposite B by b = CA, and opposite C by c = AB. The distance formula gives: c = AB = √((2 − 1)² + (3 + 2)²) = √(1 + 25) = √26 ≈ 5.099; a = BC = √((6 − 2)² + (5 − 3)²) = √(16 + 4) = √20 = 2√5 ≈ 4.472; b = CA = √((6 − 1)² + (5 + 2)²) = √(25 + 49) = √74 ≈ 8.602. The longest side is b ≈ 8.602, so the largest angle is at B, and since $b^2 = 74 > a^2 + c^2 = 20 + 26 = 46$, the angle at B must be obtuse — a useful prediction before computing, because the cosine rule should now produce a negative cosine for B.

Part (b) continued: The three angles by the cosine rule

The cosine rule, cos A = (b² + c² − a²)/(2bc), gives cos A = (74 + 26 − 20)/(2·√74·√26) = 80/(2√1924) = 40/√1924 ≈ 0.9119, so A ≈ 24.23°. Next, cos B = (a² + c² − b²)/(2ac) = (20 + 26 − 74)/(2·√20·√26) = −28/(2√520) = −14/√520 ≈ −0.6139, so B ≈ 127.87° — obtuse, exactly as predicted. Finally, cos C = (a² + b² − c²)/(2ab) = (20 + 74 − 26)/(2·√20·√74) = 68/(2√1480) = 34/√1480 ≈ 0.8838, so C ≈ 27.90°. The three angles are collected in the table, together with the check that they sum to 180°.

Angle	Cosine (exact)	Cosine (decimal)	Measure
A	40/√1924	0.9119	≈ 24.23°
B	−14/√520	−0.6139	≈ 127.87°
C	34/√1480	0.8838	≈ 27.90°
Sum	—	—	180.00° ✓

Part (b) continued: The area

The coordinate (shoelace) formula gives the area directly from the vertices: Area = (1/2) | x_A(y_B − y_C) + x_B(y_C − y_A) + x_C(y_A − y_B) | = (1/2) | 1(3 − 5) + 2(5 − (−2)) + 6(−2 − 3) | = (1/2) | −2 + 14 − 30 | = (1/2)(18) = 9 square units. Cross-check with trigonometry: Area = (1/2)·c·b·sin A = (1/2)(√26)(√74) sin(24.23°) ≈ (1/2)(43.86)(0.4104) ≈ 9.00, in agreement. A second cross-check uses the obtuse angle: (1/2)·a·c·sin B = (1/2)(√20)(√26) sin(127.87°) ≈ (1/2)(22.80)(0.7894) ≈ 9.00. Three independent computations give the same area, so the result is secure.

Area = 9 square units

Remarks: an alternative route and the shape of the triangle

Two closing observations consolidate the methods of this question. First, part (a) can also be solved by the general-circle method: assume the circle through the three vertices has equation $x^2 + y^2 + 2gx + 2fy + c = 0$, substitute A, B and C to obtain three linear equations in g, f and c, namely $14g - 4f + c = -53$, $-4g + 2f + c = -5$ and $6g + 8f + c = -25$, and solve them simultaneously; the centre $(-g, -f)$ comes out as $(17/7, -5/7)$ (with $c = -113/7$) and the radius $\sqrt{(g^2 + f^2 - c)} = \sqrt{1105/7}$, in complete agreement with the perpendicular-bisector method — a reassuring instance of two different methods certifying one answer. Secondly, the obtuse angle found in part (b) explains a geometric subtlety connecting the two parts: for an obtuse triangle the circumcenter lies OUTSIDE the triangle (on the far side of the longest side), whereas for an acute triangle it lies inside and for a right triangle it is the midpoint of the hypotenuse. In part (a) the triangle's squared sides are $AB^2 = 90$, $BC^2 = 34$ and $CA^2 = 52$; since the largest, 90, exceeds the sum of the other two, $34 + 52 = 86$, the angle at C is (slightly) obtuse, and consistently the circumcenter $(17/7, -5/7)$ sits just outside the triangle, across side AB — indeed, substituting O into the equation $x + 3y - 1 = 0$ of line AB gives $-5/7$, while C gives $+14$, so O and C lie on opposite sides of AB, as the figure shows. Students can use the test 'largest squared side versus sum of the other two squared sides' as a one-line predictor of this behaviour before computing anything.

Conclusion

Part (a): the circumcenter is $O(17/7, -5/7) \approx (2.43, -0.71)$, with circumradius $\sqrt{1105/7} \approx 4.75$, verified by three equal squared distances $1105/49$. Part (b): the sides are $\sqrt{26}$, $\sqrt{20}$ and $\sqrt{74}$; the interior angles are $A \approx 24.23°$, $B \approx 127.87°$ (obtuse, as the side lengths predicted) and $C \approx 27.90°$,

summing to 180°; and the area is exactly 9 square units, confirmed by two trigonometric cross-checks.

Question No. 5 [20] — (a) Graph the feasible region of the following system and find the corner points: −2x + 5y ≤ 150; 3x + 5y ≤ 120; x − 7y ≤ 77. (b) Maximize the function defined by f(x, y) = 320x + 210y subject to constraints: 2x + 3y ≤ 300; 3x + 5y ≤ 500; x ≥ 0; y ≥ 0.

Introduction

This question is about linear programming in two variables. A linear inequality in x and y describes a closed half-plane whose boundary is a straight line; a system of such inequalities describes the intersection of the half-planes, called the feasible region. When the region is bounded it is a convex polygon, and its vertices are called corner points. The fundamental theorem of the subject says that a linear objective function attains its maximum and minimum over such a region at corner points, so the practical method is: graph the region, find the corners by solving pairs of boundary equations, and evaluate. Part (a) asks only for the region and its corners; part (b) adds an objective function to be maximized. One feature of part (a) deserves note: unlike part (b), the printed system in part (a) contains no conditions $x \geq 0$, $y \geq 0$, so the feasible region is not confined to the first quadrant — it turns out to be a triangle straddling three quadrants.

Part (a): The three boundary lines

Replace each inequality by an equation. Line L_1: $-2x + 5y = 150$ passes through (0, 30) and (−75, 0); the origin satisfies $-0 + 0 \leq 150$, so the feasible side is the side containing the origin. Line L_2: $3x + 5y = 120$ passes through (0, 24) and (40, 0); the origin satisfies $0 \leq 120$, so again the origin side is feasible. Line L_3: $x - 7y = 77$ passes through (77, 0) and (0, −11); the origin satisfies $0 \leq 77$,

so the origin side is feasible. The feasible region is therefore the set of points that lie simultaneously on the origin side of all three lines. Because the three lines are in 'general position' (no two parallel, not concurrent), they form a triangle, and we now check that the triangle they form is exactly the feasible region: each vertex of the triangle must satisfy the remaining inequality, and the interior near the centroid must satisfy all three, which it does since the centroid of the vertices computed below is (approximately) (−39.4, −3.5) and substituting gives −2(−39.4) + 5(−3.5) ≈ 61.3 ≤ 150, 3(−39.4) + 5(−3.5) ≈ −135.7 ≤ 120 and −39.4 − 7(−3.5) ≈ −14.9 ≤ 77. Hence the region is bounded — a triangle — even without non-negativity conditions.

Part (a) continued: The corner points

Each corner is the intersection of two boundary lines, checked against the third inequality. Corner 1, $L_1 \cap L_2$: subtracting the equations, $(-2x + 5y) - (3x + 5y) = 150 - 120$ gives $-5x = 30$, so $x = -6$; then $-2(-6) + 5y = 150$ gives $5y = 138$, $y = 138/5 = 27.6$. Check L_3: $-6 - 7(27.6) = -199.2 \leq 77$ ✓ Corner 2, $L_1 \cap L_3$: from L_3, $x = 77 + 7y$; substituting in L_1, $-2(77 + 7y) + 5y = 150$ gives $-154 - 9y = 150$, so $y = -304/9 \approx -33.78$ and $x = 77 + 7(-304/9) = -1435/9 \approx -159.44$. Check L_2: $3(-159.44) + 5(-33.78) \approx -647.2 \leq 120$ ✓ Corner 3, $L_2 \cap L_3$: substituting $x = 77 + 7y$ in L_2, $3(77 + 7y) + 5y = 120$ gives $231 + 26y = 120$, so $y = -111/26 \approx -4.27$ and $x = 77 + 7(-111/26) = 1225/26 \approx 47.12$. Check L_1: $-2(47.12) + 5(-4.27) \approx -115.6 \leq 150$ ✓ The corners are summarised in the table and drawn in the figure.

Corner	Lines meeting	Coordinates (exact)	Coordinates (decimal)	Third inequality
P_1	L_1 and L_2	$(-6, 138/5)$	(−6, 27.60)	$-199.2 \leq 77$ ✓

P_2	L_1 and L_3	(−1435/9, −304/9)	(−159.44, −33.78)	−647.2 ≤ 120 ✓
P_3	L_2 and L_3	(1225/26, −111/26)	(47.12, −4.27)	−115.6 ≤ 150 ✓

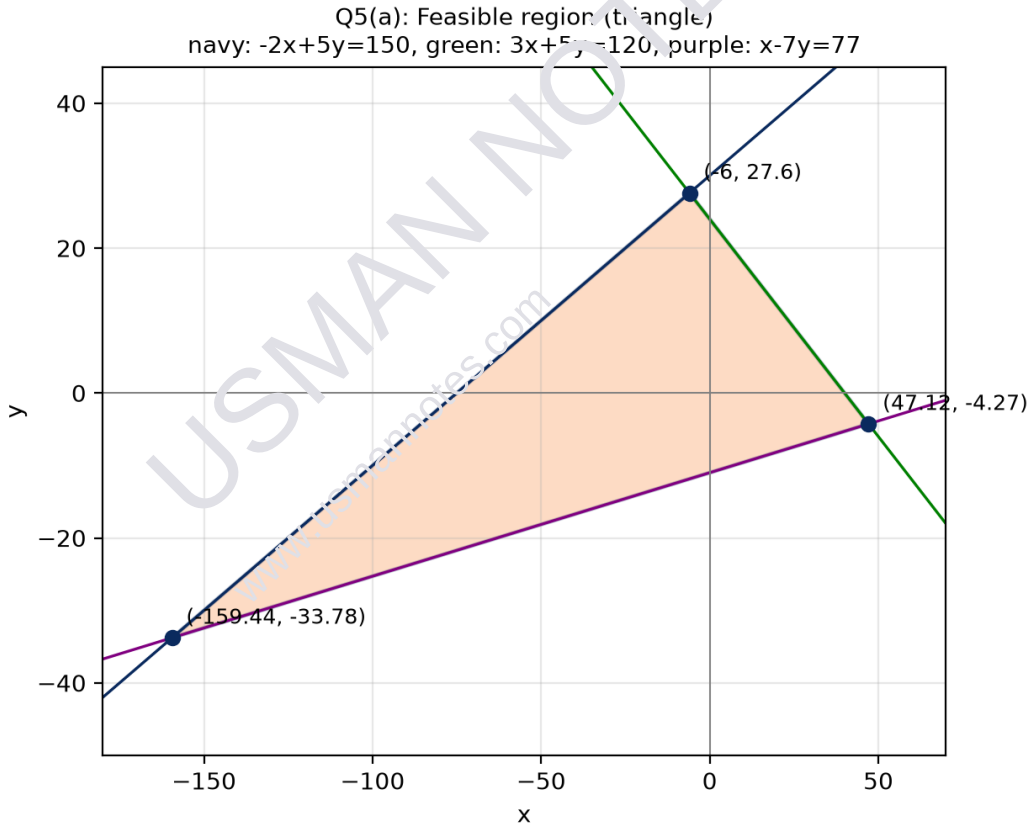


Figure: The feasible region of part (a) is the shaded triangle with the three corner points marked.

Part (b): Setting up the maximization

Now maximize f(x, y) = 320x + 210y subject to 2x + 3y ≤ 300, 3x + 5y ≤ 500, x ≥ 0, y ≥ 0. The boundary lines are 2x + 3y = 300, through (150, 0) and (0, 100), and 3x + 5y = 500, through (500/3, 0) ≈ (166.7, 0) and (0, 100). Notice immediately that both lines pass through (0, 100) — their intersection is on the y-axis — and that at x = 150 the second line gives y = (500 − 450)/5 =

10 > 0, so on the segment of the first line inside the first quadrant, the second constraint is satisfied everywhere except that it is exactly tight at (0, 100): indeed, along 2x + 3y = 300 with 0 ≤ x ≤ 150, we have 3x + 5y = 3x + (5/3)(300 − 2x) = 500 − x/3 ≤ 500. The second constraint is therefore redundant (it cuts off nothing), and the feasible region is the triangle with vertices (0, 0), (150, 0) and (0, 100). The corner-point candidates are just these three vertices.

Part (b) continued: Evaluating the objective at the corners

Corner point	How obtained	f = 320x + 210y
(0, 0)	x = 0 and y = 0	0
(150, 0)	y = 0 and 2x + 3y = 300	320(150) + 0 = 48,000
(0, 100)	x = 0; both lines meet here	0 + 210(100) = 21,000

The largest value is 48,000, attained at (x, y) = (150, 0). An iso-profit (level-line) argument confirms the geometry: the level lines 320x + 210y = k have slope −320/210 ≈ −1.524, steeper than the binding edge 2x + 3y = 300, whose slope is −2/3 ≈ −0.667; as k increases the level line therefore last touches the feasible triangle at the x-intercept (150, 0), not at (0, 100). Also confirm feasibility of the maximizer against the second constraint: 3(150) + 5(0) = 450 ≤ 500 ✓ (with slack 50), and against the first: 2(150) = 300, exactly tight. So the maximum uses the first resource completely and leaves 50 units of the second resource unused.

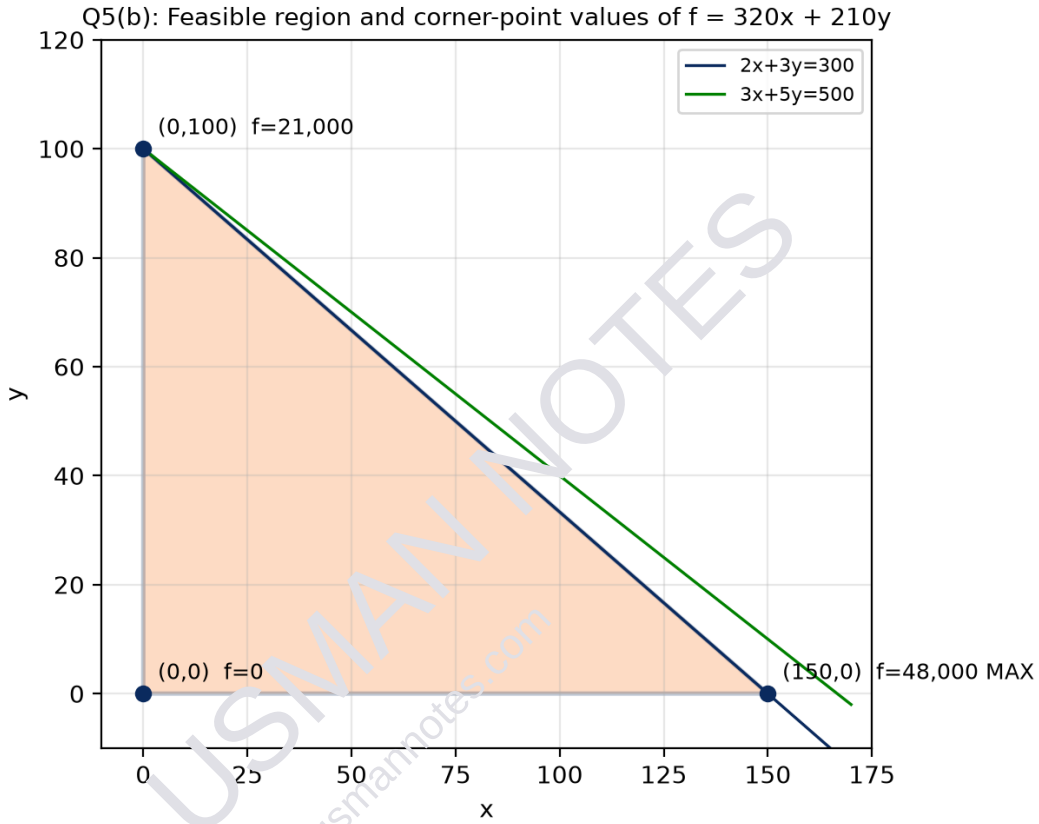


Figure: The feasible region of part (b) is the shaded triangle; f is largest at (150, 0).

Conclusion

Part (a): the feasible region is the triangle with corner points (−6, 27.6), (−1435/9, −304/9) ≈ (−159.44, −33.78) and (1225/26, −111/26) ≈ (47.12, −4.27), each verified against the remaining inequality. Part (b): the feasible region is the triangle (0, 0), (150, 0), (0, 100) — the constraint $3x + 5y \leq 500$ being redundant except at (0, 100) — and the maximum of $f(x, y) = 320x + 210y$ is 48,000, attained at (x, y) = (150, 0).