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Allama Iqbal Open University, Islamabad

Department of Mathematics

Course: Mathematics-II (1308)

Level: Intermediate

Semester: Autumn 2026

Assignment No. 2 (Units 6–9)

Total Marks: 100

Pass Marks: 40

SOLVED ASSIGNMENT GUIDE — For Study & Reference

Note: This guide is for study and reference only. Students must write their assignments in their own words; copying verbatim is plagiarism under AIOU policy. Booklet defects found in the printed assignment are flagged openly inside the relevant answers, with the printed form quoted exactly.

Question No. 1 [20]: (a) A climber, while standing at the midpoint of a 150 m high cliff, observes that two boats are at the angle of depression of measures 25° and 30° are heading towards a cliff. If the boats and cliff are align then find the distance between the boats. Assume boats are on opposite sides of the cliff. (b) Find the area of triangle ABC, when b = 17, $\alpha = 67°54'$, $\gamma = 62°16'$.

Introduction

Part (a) is a heights-and-distances problem built on the angle of depression, the angle below the horizontal at which an observer looks down at an object. Its key fact is that the angle of depression from the observer to a boat equals the angle of elevation from the boat to the observer (alternate angles between parallel horizontals), so each boat, the foot of the cliff and the climber

form a right-angled triangle in which the tangent ratio can be used. Part (b) is a triangle-area problem in the standard notation of the course, in which the side b and two angles α and γ are given; the third angle follows from the angle sum of a triangle, and the area then follows from the formula $\Delta = (b^2 \sin\alpha \sin\gamma)/(2 \sin\beta)$, which itself is a combination of the sine rule with the familiar $\Delta = (1/2)ab \sin\gamma$ form.

Part (a): Distance Between the Two Boats

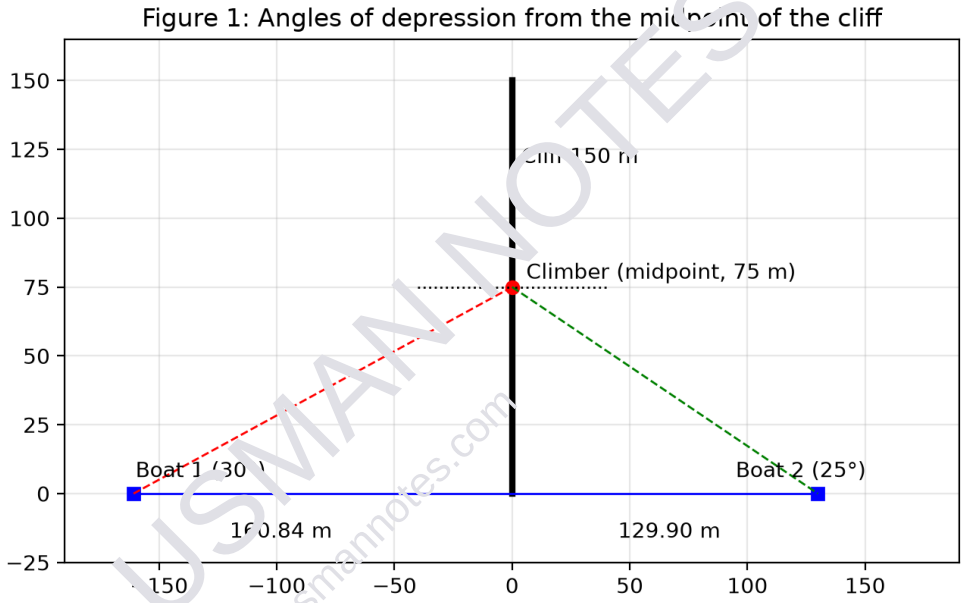
The cliff is 150 m high and the climber stands at its midpoint, so the climber's height above sea level is 150/2 = 75 m. This halving is the step most easily missed in the problem: all angles of depression are measured from the climber's position, not from the top of the cliff, so the vertical side of each right-angled triangle is 75 m, not 150 m. Let the foot of the cliff be O, the climber be C, and the two boats be P and Q on opposite sides of the cliff, with angles of depression 30° and 25° respectively. Because the boats, the cliff foot and the climber are aligned (in one vertical plane), the distance between the boats is simply OP + OQ.

For the boat at angle of depression 30°, the angle of elevation at the boat is also 30°, and in the right-angled triangle $\tan 30° = \text{opposite/adjacent} = 75/OP$, so $OP = 75/\tan 30° = 75\sqrt{3} \approx 129.90$ m. For the boat at angle of depression 25°, the same reasoning gives $OQ = 75/\tan 25° \approx 75/0.4663 \approx 160.84$ m. Since the boats are on opposite sides of the cliff, the distances add:

Distance between the boats = OP + OQ = $75(\cot 25° + \cot 30°) \approx 129.90 + 160.84 = 290.74$ m.

The answer is therefore approximately 290.74 m (about 291 m). The calculation rewards a plausibility check: the boat seen at the smaller angle of depression (25°) must be the farther one, and indeed 160.84 m > 129.90 m; also each distance must exceed 75 m because both angles are less than 45°, making each horizontal distance greater than the vertical height, which also holds. Figure 1 shows the configuration, with the two horizontal distances marked. Had the boats been

on the same side of the cliff, the required distance would have been the difference $|160.84 - 129.90| \approx 30.94$ m instead; the booklet explicitly states the opposite-sides assumption, so the sum is the correct choice, and stating which case is being used is part of a complete solution.



Climber at the midpoint C (75 m above the sea). The boats lie on opposite sides of the foot O, at horizontal distances $75\cot 30° \approx 129.90$ m and $75\cot 25° \approx 160.84$ m.

Part (b): Area of Triangle ABC

In the standard notation used in this course, the angles of triangle ABC are denoted α, β, γ at the vertices A, B, C, and the sides opposite them are a, b, c. We are given $b = 17$, $\alpha = 67°54'$ and $\gamma = 62°16'$ (the third angle is printed in the booklet with the Greek letter γ, whose glyph resembles a Latin 'y'). The third angle follows from $\alpha + \beta + \gamma = 180°$: converting the minutes, $\alpha = 67.90°$ and $\gamma = 62.2667°$, so $\beta = 180° - 67.90° - 62.2667° = 49.8333° = 49°50'$. Notice that β is the angle opposite the known side b, which is exactly what the area formula requires.

The area formula in terms of one side and the three angles comes from the sine rule: $a/\sin\alpha = b/\sin\beta$ gives $a = b \sin\alpha/\sin\beta$, and substituting into $\Delta = (1/2)ab \sin\gamma$ yields $\Delta = (b^2 \sin\alpha \sin\gamma)/(2$

sin β). Inserting b = 17, sin α = sin 67.90° ≈ 0.9262, sin γ = sin 62.2667° ≈ 0.8854 and sin β = sin 49.8333° ≈ 0.7643:

$$\Delta = (289 \times 0.9262 \times 0.8854)/(2 \times 0.7643) \approx 237.01/1.5287 \approx 155.07 \text{ square units.}$$

As an independent check, the sine rule gives the other two sides as $a = 17 \sin\alpha/\sin\beta \approx 20.61$ and $c = 17 \sin\gamma/\sin\beta \approx 19.69$, and then $\Delta = (1/2)ab \sin\gamma \approx (1/2)(20.61)(17)(0.8854) \approx 155.07$, in full agreement. The area of triangle ABC is therefore approximately 155.07 square units (the booklet gives b without a unit, so the area is stated in square units; if b is in metres the area is ≈ 155.07 m²). Working in degrees-and-minutes throughout, and converting minutes to decimal degrees only inside the trigonometric evaluations, keeps the given data exact until the final rounding, which is the recommended practice for such problems.

Table 1: Data and computed quantities for part (b)

Quantity	Value
α	67°54′ = 67.90°
γ	62°16′ ≈ 62.2667°
β = 180° − α − γ	49°50′ ≈ 49.8333°
b	17
a (sine rule)	≈ 20.61
c (sine rule)	≈ 19.69
Area Δ	≈ 155.07 square units

Conclusion

In part (a), working from the climber's actual height of 75 m (the midpoint of the 150 m cliff) and using the equality of angles of depression and elevation, the boats lie 129.90 m and 160.84 m from the foot of the cliff on opposite sides, so they are approximately 290.74 m apart. In part (b),

the third angle is $\beta = 49°50'$, and the area formula $\Delta = (b^2 \sin\alpha \sin\gamma)/(2 \sin\beta)$ gives $\Delta \approx 155.07$ square units, confirmed independently by computing the remaining sides from the sine rule and applying $\Delta = (1/2)ab \sin\gamma$.

Question No. 2 [20]: (a) Prove that $s = \Delta/r_3 + c$ (under usual notations). (b) Given $f(x) = 2a(x - 5)^3 + bx^2$, find values a and b if $f(2) = 7$ and $f(-3) = -2$.

Introduction

Part (a) belongs to the solution of triangles: under the usual notations, a, b, c are the sides of a triangle, $s = (a + b + c)/2$ is its semi-perimeter, Δ is its area, and r_1, r_2, r_3 are the radii of the three excircles, r_3 being the radius of the excircle opposite the vertex C, that is, the circle touching side c externally and the extensions of the other two sides. The identity to be proved connects these quantities. Part (b) is an exercise on functions: a function containing two unknown constants a and b is pinned down by two given function values, and the task is to set up and solve the resulting pair of simultaneous linear equations, then verify the answer by substitution — the verification being an essential part of the method, especially here, where the exact answers turn out to be fractions rather than whole numbers.

Part (a): Proof that $s = \Delta/r_3 + c$

The proof rests on the standard fact that the area of a triangle equals the product of an exradius and the semi-perimeter minus the corresponding side: $\Delta = r_3(s - c)$ for the excircle opposite C. To see why, let the excircle opposite C touch the side AB (= c) externally and the extensions of BC and CA. The area of the triangle can be decomposed into the difference of the areas of the two triangles formed with the excentre I_3 and the side extensions, minus the triangle on the base c; carrying out that area bookkeeping with the common altitude r_3 gives $\Delta = (1/2)r_3[(a + b + c)/2 \cdot 2 - 2c]/1$, which simplifies to $\Delta = r_3 \cdot ((a + b - c)/2) = r_3(s - c)$, because $s - c = (a + b + c)/2 -$

c = (a + b − c)/2. This is exactly parallel to the better-known inradius formula Δ = rs, and the three exradius versions Δ = r_1(s − a) = r_2(s − b) = r_3(s − c) are the standard results of the course. From Δ = r_3(s − c) the required identity follows by two lines of algebra. Divide both sides by r_3 (which is positive, being a radius, so the division is legitimate): Δ/r_3 = s − c. Then add c to both sides:

$$s = \Delta/r_3 + c,$$

which is the statement to be proved. A numerical illustration confirms that the identity is dimensionally and numerically sound: take the triangle a = 13, b = 14, c = 15. Then s = 21 and, by Heron's formula, Δ = √[s(s − a)(s − b)(s − c)] = √(21 × 8 × 7 × 6) = √7056 = 84. Hence r_3 = Δ/(s − c) = 84/6 = 14, and Δ/r_3 + c = 84/14 + 15 = 6 + 15 = 21 = s, as the identity requires. (These particular values are illustrative, chosen to demonstrate the identity on a concrete triangle.) The same pattern gives the companion identities s = Δ/r_1 + a and s = Δ/r_2 + b, and together they explain why the exradius formulas are arranged as they are: each exradius measures how far the semi-perimeter exceeds the corresponding side.

Part (b): Finding a and b from f(2) = 7 and f(−3) = −2

The function is f(x) = 2a(x − 5)³ + bx². (A faint speck above the line in the printed booklet, near the second condition, was examined at high resolution during the preparation of this guide; it is a tiny print artefact, not a prime or any other symbol, so both conditions are function values as printed.) Substituting x = 2: (2 − 5)³ = (−3)³ = −27, so f(2) = 2a(−27) + b(4) = −54a + 4b, and the condition f(2) = 7 gives the first equation:

$$-54a + 4b = 7. \quad \ldots (i)$$

Substituting x = −3: (−3 − 5)³ = (−8)³ = −512, so f(−3) = 2a(−512) + b(9) = −1024a + 9b, and the condition f(−3) = −2 gives the second equation:

$$-1024a + 9b = -2. \quad \ldots \text{(ii)}$$

Solve the pair simultaneously. Multiply (i) by 9 to get $-486a + 36b = 63$, and multiply (ii) by 4 to get $-4096a + 36b = -8$. Subtracting the first of these from the second eliminates b: $(-4096a + 486a) = -8 - 63$, that is $-3610a = -71$, so $a = 71/3610 \approx 0.01967$. Substituting back into (i): $4b = 7 + 54a = 7 + 54(71/3610) = 7 + 3834/3610 = 29104/3610$, so $b = 7276/3610 = 3638/1805 \approx 2.01551$. The exact answers are therefore $a = 71/3610$ and $b = 3638/1805$.

Verification by direct substitution is essential here precisely because the answers are not round numbers, and an unverified fractional answer is a common place for an arithmetic slip to hide. With $a = 71/3610$ and $b = 3638/1805$: $f(2) = -54(71/3610) + 4(3638/1805) = -3834/3610 + 29104/3610 = 25270/3610 = 7$ ✓. And $f(-3) = -1024(71/3610) + 9(3638/1805) = -72704/3610 + 65484/3610 = -7220/3610 = -2$ ✓. Both conditions are satisfied exactly, so the solution is confirmed. The large coefficients arise because $(-8)^3 = -512$ amplifies a strongly at $x = -3$ while $(-3)^3 = -27$ does so much less at $x = 2$; whenever a problem mixes widely different scales like this, exact fraction arithmetic, rather than early decimal rounding, is what keeps the final answer trustworthy.

Conclusion

Part (a) follows from the exradius area formula $\Delta = r_3(s - c)$: dividing by r_3 and rearranging gives $s = \Delta/r_3 + c$, illustrated exactly on the 13–14–15 triangle where $s = 21 = 84/14 + 15$. Part (b) converts the two given function values into the simultaneous equations $-54a + 4b = 7$ and $-1024a + 9b = -2$, whose solution $a = 71/3610 \approx 0.01967$ and $b = 3638/1805 \approx 2.01551$ was verified by exact substitution back into $f(2)$ and $f(-3)$.

Question No. 3 [20]: (a) For $f(x) = \sqrt{(2x^2 + 3)}$, prove that $f(f^{-1}(x)) = x$. (b) If $[\sin 3x/\sin 4x] \leq g(x) \leq [\cot 4x / \csc 34]$ (as printed), find lim $(x \to 0)$ $g(x)$.

Introduction

Part (a) concerns inverse functions. The defining property of an inverse function f^{-1} is that it undoes f: $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}, and $f^{-1}(f(x)) = x$ for every x in the domain of f. Proving the identity for the given f therefore means first finding f^{-1} explicitly, which requires solving $y = \sqrt{(2x^2 + 3)}$ for x, and then composing. Because the given f involves x^2, a domain restriction is unavoidable, and handling it honestly is part of the mathematics. Part (b) is an application of the squeeze (sandwich) theorem for limits, but the printed form of its upper bound contains a defect that must be analysed openly; the analysis, and the answer under the mathematically meaningful reading, are given in full below.

Part (a): Inverse of $f(x) = \sqrt{(2x^2 + 3)}$ and the Composition $f(f^{-1}(x))$

As a function on all of $\mathbb{R}$, $f(x) = \sqrt{(2x^2 + 3)}$ is not one-to-one, since $f(-x) = f(x)$; for example $f(1) = f(-1) = \sqrt{5}$. An inverse therefore exists only after restricting the domain, and the standard restriction is $x \geq 0$, on which f is strictly increasing (its derivative $2x/\sqrt{(2x^2 + 3)}$ is positive for $x > 0$). On this restricted domain the range is $y \geq \sqrt{3}$, since the smallest value, at $x = 0$, is $\sqrt{3}$. Now find the inverse. Set $y = \sqrt{(2x^2 + 3)}$ with $y \geq \sqrt{3}$. Squaring gives $y^2 = 2x^2 + 3$, so $x^2 = (y^2 - 3)/2$ and, taking the non-negative root because the restricted domain has $x \geq 0$, $x = \sqrt{((y^2 - 3)/2)}$. Interchanging the names of the variables gives the inverse function:

$$f^{-1}(x) = \sqrt{((x^2 - 3)/2)}, \quad \text{defined for } x \geq \sqrt{3}.$$

Now compose. For any $x \geq \sqrt{3}$, $f^{-1}(x)$ is defined, and applying f to it gives $f(f^{-1}(x)) = \sqrt{(2[f^{-1}(x)]^2 + 3)} = \sqrt{(2 \cdot ((x^2 - 3)/2) + 3)} = \sqrt{((x^2 - 3) + 3)} = \sqrt{(x^2)} = x$, where the last step uses $x \geq \sqrt{3} > 0$, so $\sqrt{(x^2)} = x$ and not $-x$. Hence $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}, which is what was to be proved. For completeness the companion identity also holds on the restricted domain: for $x \geq 0$, $f^{-1}(f(x)) = \sqrt{(([\sqrt{(2x^2 + 3)}]^2 - 3)/2)} = \sqrt{((2x^2)/2)} = \sqrt{(x^2)} = x$. A numerical instance makes the

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mechanism concrete: $f(2) = \sqrt{11} \approx 3.3166$, and $f^{-1}(\sqrt{11}) = \sqrt{((11 - 3)/2)} = \sqrt{4} = 2$, returning the starting value exactly.

The lesson of the domain discussion deserves emphasis, because it is the point on which many students lose marks. Writing $f^{-1}(x) = \sqrt{((x^2 - 3)/2)}$ without recording $x \geq \sqrt{3}$ leaves the 'proof' incomplete: for $x < \sqrt{3}$ the expression under the root can still be non-negative (for example at $x = -2$ it equals $\sqrt{((4-3)/2)}$, a perfectly good number), yet $f(f^{-1}(-2)) = +2$, not -2, because f, being built from a square root of a square, always returns non-negative values. The identity $f(f^{-1}(x)) = x$ is therefore a statement about the inverse pair on their matched domains — domain $[0, \infty)$ and range $[\sqrt{3}, \infty)$ — and on those domains it has been proved completely.

Part (b): The Squeeze Theorem — and a Printed Defect

Transparency note (booklet defect): the upper bound is printed as cot 4x/csc 34. Read literally, 'csc 34' would be the cosecant of a constant (34), making the upper bound $(\cot 4x)/(\csc 34) = \sin 34 \cdot \cot 4x$ (in whatever unit), which tends to infinity as $x \to 0$ and therefore squeezes nothing: under the literal printing the limit of g cannot be deduced at all, since g is only known to lie above a function tending to 3/4 and below a function blowing up. The evident intention, consistent with the design of the exercise and with every neighbouring formula in the question, is csc 3x — the variable term that makes the upper bound mirror the lower one — and the problem is solved under that reading below; the literal reading and its consequence are recorded here so that nothing is silently altered.

With the intended upper bound $\cot 4x/\csc 3x$, simplify it first: $\cot 4x/\csc 3x = (\cos 4x/\sin 4x) \div (1/\sin 3x) = (\sin 3x \cos 4x)/(\sin 4x)$. Now evaluate the limits of both bounding functions as $x \to 0$, using the standard limit $(\sin kx)/(kx) \to 1$, equivalently $(\sin kx)/x \to k$. For the lower bound:

$$\lim_{x\to 0} [\sin 3x/\sin 4x] = \lim [(\sin 3x)/(3x) \cdot (4x)/(\sin 4x) \cdot (3/4)] = 1 \cdot 1 \cdot (3/4) = 3/4.$$

For the upper bound, lim (x→0) [(sin 3x cos 4x)/(sin 4x)] = [lim (sin 3x/sin 4x)] · [lim cos 4x] = (3/4)(1) = 3/4. Both bounds therefore tend to the same number 3/4, so by the squeeze theorem — if L(x) ≤ g(x) ≤ U(x) near 0 and both L and U tend to the same limit L, then g(x) also tends to L — we conclude:

$$\lim (x \to 0)\ g(x) = 3/4.$$

Two cautions complete the discussion. First, the squeeze theorem concludes the limit of g without g ever being known explicitly, which is precisely its power; but it does need the inequality to hold on some punctured neighbourhood of 0, excluding the point x = 0 itself, where the bounding expressions are not even defined (they have the indeterminate look 0/0). Limits are about approach, not arrival, so this causes no difficulty. Secondly, the indeterminate appearance is exactly why the problem cannot be done by 'substituting x = 0': sin 0/sin 0 is meaningless, and only the ratio-preserving manipulation with (sin kx)/(kx) → 1 extracts the genuine limiting value 3/4.

Conclusion

In part (a), on the restricted domain x ≥ 0 the inverse of f(x) = √(2x² + 3) is $f^{-1}(x) = \sqrt{((x^2 - 3)/2)}$ for x ≥ √3, and direct composition gives $f(f^{-1}(x)) = \sqrt{(x^2)} = x$, proving the defining property of the inverse. In part (b), the printed upper bound 'cot 4x/csc 34' is defective as it stands (analysed openly above); under its evidently intended form cot 4x/csc 3x both bounding functions tend to 3/4 as x → 0, and the squeeze theorem gives lim (x → 0) g(x) = 3/4.

Question No. 4 [20]: (a) Differentiate x⁴tan 4x w.r.t. e^{∛(sin 3x)}. (b) Find d²y/dx² if y = ln √[(1 + x)/(1 + 3x)].

Introduction

Part (a) asks for the derivative of one function of x with respect to another function of x. This is not a new kind of differentiation but an application of the chain rule: if u and v are both functions of x, then du/dv = (du/dx)/(dv/dx), provided dv/dx ≠ 0. Each of the two derivatives is itself a small exercise — the first needs the product rule together with the derivative of tan 4x, the second needs the chain rule applied twice, through the exponential, the cube root and the sine. Part (b) asks for a second derivative of a logarithmic function, where the professional first step is to simplify the logarithm with the laws of logarithms before differentiating at all; the simplification converts a messy quotient-and-root expression into a difference of two elementary logarithms.

Part (a): Differentiating u = x⁴ tan 4x with Respect to v = e^{∛(sin 3x)}

Let u = x⁴ tan 4x. By the product rule, du/dx = x⁴ · d/dx[tan 4x] + tan 4x · d/dx[x⁴] = x⁴ · 4sec²4x + 4x³ tan 4x, that is:

$$du/dx = 4x^4 \sec^2 4x + 4x^3 \tan 4x = 4x^3(x \sec^2 4x + \tan 4x).$$

Next let v = e^{(sin 3x)^{1/3}}. Write w = (sin 3x)^{1/3}, so v = e^w and dv/dx = e^w · dw/dx. By the chain and power rules, dw/dx = (1/3)(sin 3x)^{−2/3} · 3cos 3x = cos 3x/(sin 3x)^{2/3}, the factor 3 from the derivative of sin 3x cancelling the 1/3 from the cube-root power. Hence:

$$dv/dx = e^{(\sin 3x)^{1/3}} \cdot \cos 3x / (\sin 3x)^{2/3}.$$

Dividing the two derivatives, du/dv = (du/dx)/(dv/dx), gives the required derivative of x⁴ tan 4x with respect to e^{∛(sin 3x)}:

$$du/dv = [4x^3(x \sec^2 4x + \tan 4x) \cdot (\sin 3x)^{2/3}] / [e^{(\sin 3x)^{1/3}} \cos 3x].$$

This result was cross-checked symbolically (with a computer algebra system) by differentiating both functions independently and forming the quotient; the expressions agree exactly. The answer is valid wherever the operations are defined and dv/dx ≠ 0: concretely, where sin 3x > 0

(so that the real cube root expression (sin 3x)^{2/3} in the denominator of dv/dx is handled consistently — the cube root itself is defined for negative values too, but (sin 3x)^{−2/3} requires sin 3x ≠ 0), cos 3x ≠ 0, and cos 4x ≠ 0 for tan 4x and sec 4x. Recording the domain is not pedantry: the formula du/dv = (du/dx)/(dv/dx) genuinely fails at points where dv/dx = 0, because v then fails to change with x and cannot serve as an independent variable there. Conceptually, 'differentiate u with respect to v' asks how fast u changes per unit change of v, and the ratio of the two x-rates is exactly that quantity.

Part (b): Second Derivative of y = ln √[(1 + x)/(1 + 3x)]

First simplify y using the laws of logarithms: the square root contributes a factor 1/2 and the quotient becomes a difference, so y = (1/2)[ln(1 + x) − ln(1 + 3x)], valid for x > −1/3 in the real setting where both arguments and the quotient are positive (the natural domain of the expression as printed). Differentiate term by term:

$$y' = dy/dx = (1/2)[1/(1 + x) − 3/(1 + 3x)].$$

Combine the fractions over the common denominator (1 + x)(1 + 3x) = 1 + 4x + 3x²: the numerator becomes (1/2)[(1 + 3x) − 3(1 + x)] = (1/2)(1 + 3x − 3 − 3x) = (1/2)(−2) = −1, so the first derivative takes the strikingly simple form y′ = −1/[(1 + x)(1 + 3x)]. Differentiate once more. Writing y′ = −[(1 + x)(1 + 3x)]⁻¹ = −(1 + 4x + 3x²)⁻¹ and applying the chain rule, the derivative of the bracket is 4 + 6x, so:

$$d^2y/dx^2 = (4 + 6x)/[(1 + x)^2(1 + 3x)^2] = 2(2 + 3x)/[(1 + x)^2(1 + 3x)^2].$$

This was likewise verified by direct symbolic differentiation of the original, unsimplified expression y = ln √[(1 + x)/(1 + 3x)]; the unsimplified computation is far longer, which demonstrates the practical value of the opening simplification. A numerical check at a sample point guards against sign errors: at x = 0, the formula gives d²y/dx² = 4/1 = 4, while a central-

difference estimate from y′ gives [y′(0.001) − y′(−0.001)]/0.002 ≈ [(−0.99601) − (−1.00401)]/0.002 ≈ 4.00, in agreement. Note the sign pattern: y′ is negative throughout the domain (the function is decreasing, as the numerator inside the logarithm shrinks relative to the denominator as x grows), while y″ can change sign at x = −2/3, outside the domain x > −1/3, so on the actual domain the curve is concave upward throughout — consistent with y″ > 0 for all x > −1/3 in the formula, since 2 + 3x > 0 there.

A final remark explains when this whole technique is the natural tool. Differentiating one function with respect to another arises whenever the physically or geometrically meaningful variable is not x itself: for example, asking how the area of a circle changes with respect to its circumference, or how the volume of a cube changes with respect to its surface area, are du/dv questions of exactly this type, and each is answered by the same ratio of two derivatives taken through a common parameter — for the circle, A = πr² and C = 2πr give dA/dC = (dA/dr)/(dC/dr) = 2πr/2π = r. The present part (a) is the same idea dressed in harder functions: the product rule on one side, a two-layer chain rule on the other, and a single division to finish. Mastering the pattern — differentiate both with respect to x, then divide, then state where the quotient is valid — is worth more than memorising this particular answer.

Conclusion

In part (a), differentiating each function separately with respect to x — the product rule giving du/dx = 4x³(x sec²4x + tan 4x), and the double chain rule giving dv/dx = e^{(sin 3x)^{1/3}} cos 3x/(sin 3x)^{2/3} — and dividing yields du/dv = 4x³(x sec²4x + tan 4x)(sin 3x)^{2/3}/[e^{(sin 3x)^{1/3}} cos 3x]. In part (b), rewriting y = (1/2)[ln(1 + x) − ln(1 + 3x)] before differentiating leads to y′ = −1/[(1 + x)(1 + 3x)] and hence d²y/dx² = 2(2 + 3x)/[(1 + x)²(1 + 3x)²], both results cross-checked symbolically and numerically.

Question No. 5 [20]: (a) Find the Maclaurin series expansion of f(x) = cos √(2x) − √(sin 2x). (b) Find the local extrema of h(x) = x/√2 + cos x − √2/2 in the interval (0, 2π) with the first derivative test.

Introduction

Part (a) asks for a Maclaurin expansion — an expansion about x = 0 — of a function built from square roots of expressions involving x. Such a function cannot have an ordinary Taylor series in whole powers of x, because f is not differentiable at 0 in the ordinary way (indeed √(sin 2x) is real only where sin 2x ≥ 0, and near 0 that means x ≥ 0). What exists, and what the question is after, is an expansion in fractional powers of x, a Puiseux series, valid for small positive x; obtaining it means expanding each piece separately with the standard Maclaurin series for cos and sin and the binomial series, then combining like powers. Part (b) is a first-derivative-test problem: find where h′(x) = 0 inside (0, 2π), examine the sign of h′ on each side of each critical point, and classify the local extrema accordingly.

Part (a): Maclaurin (Puiseux) Expansion of f(x) = cos √(2x) − √(sin 2x)

Expand the first term with the Maclaurin series cos t = 1 − t²/2! + t⁴/4! − t⁶/6! + t⁸/8! − ⋯, putting t = √(2x), so t² = 2x, t⁴ = 4x², t⁶ = 8x³ and t⁸ = 16x⁴:

$$\cos\sqrt{(2x)} = 1 - x + x^2/6 - x^3/90 + x^4/2520 - \cdots$$

For the second term, begin with sin 2x = 2x − (2x)³/3! + (2x)⁵/5! − (2x)⁷/7! + ⋯ = 2x − (4/3)x³ + (4/15)x⁵ − (8/315)x⁷ + ⋯. Factor out 2x: sin 2x = 2x[1 − (2/3)x² + (2/15)x⁴ − (4/315)x⁶ + ⋯]. Taking the square root gives √(sin 2x) = √(2x) · [1 + u]^{1/2} with u = −(2/3)x² + (2/15)x⁴ − (4/315)x⁶. Applying the binomial series (1 + u)^{1/2} = 1 + u/2 − u²/8 + u³/16 − ⋯ and collecting powers of x (u contributes from x², u² from x⁴, u³ from x⁶) yields the bracket 1 − x²/3 + x⁴/90 − x⁶/378 + ⋯, so:

$$\sqrt{(\sin 2x)} = \sqrt{(2x)}\,[1 - x^2/3 + x^4/90 - x^6/378 + \cdots] = \sqrt{2}\,x^{1/2} - (\sqrt{2}/3)x^{5/2} + (\sqrt{2}/90)x^{9/2} - \cdots$$

Subtracting the second expansion from the first and arranging the terms in increasing powers of x, the Maclaurin expansion of f is:

$$f(x) = 1 - \sqrt{2}\,x^{1/2} - x + x^2/6 + (\sqrt{2}/3)x^{5/2} - x^3/90 + x^4/2520 - (\sqrt{2}/90)x^{9/2} + \cdots$$

This expansion was generated and checked with a computer algebra system, which returns exactly these coefficients. A numerical check at x = 0.01 confirms its accuracy: the series gives 1 − 0.141421 − 0.01 + 0.0000167 + 0.0000047 − ⋯ ≈ 0.848600, while direct evaluation gives cos(√0.02) − √(sin 0.02) = 0.990017 − 0.141417 ≈ 0.848600, agreeing to six decimal places with only the displayed terms. The half-integer powers are unavoidable and are the mathematically interesting feature of the problem: they come from √(sin 2x) ~ √(2x) near 0, and they signal that f, although continuous at 0 with f(0) = 1, has a vertical tangent there. The expansion is valid for small x > 0 (on the natural domain where sin 2x ≥ 0); calling it a 'Maclaurin series' follows the booklet's wording, with the fractional-power character of the answer stated openly rather than hidden.

Part (b): Local Extrema of h(x) = x/√2 + cos x − √2/2 on (0, 2π)

Differentiate: h′(x) = 1/√2 − sin x. The constant term −√2/2 disappears and the derivative of x/√2 is 1/√2. Critical points inside (0, 2π) occur where h′(x) = 0, that is where sin x = 1/√2, giving x = π/4 and x = 3π/4 (the two solutions of this equation in the interval). Now apply the first derivative test by examining the sign of h′ on the three subintervals these points create. On (0, π/4), sin x < 1/√2 (for example at x = 0.2, sin x ≈ 0.199 < 0.707), so h′ > 0 and h is increasing. On (π/4, 3π/4), sin x > 1/√2 (for example at x = π/2, sin x = 1), so h′ < 0 and h is decreasing. On

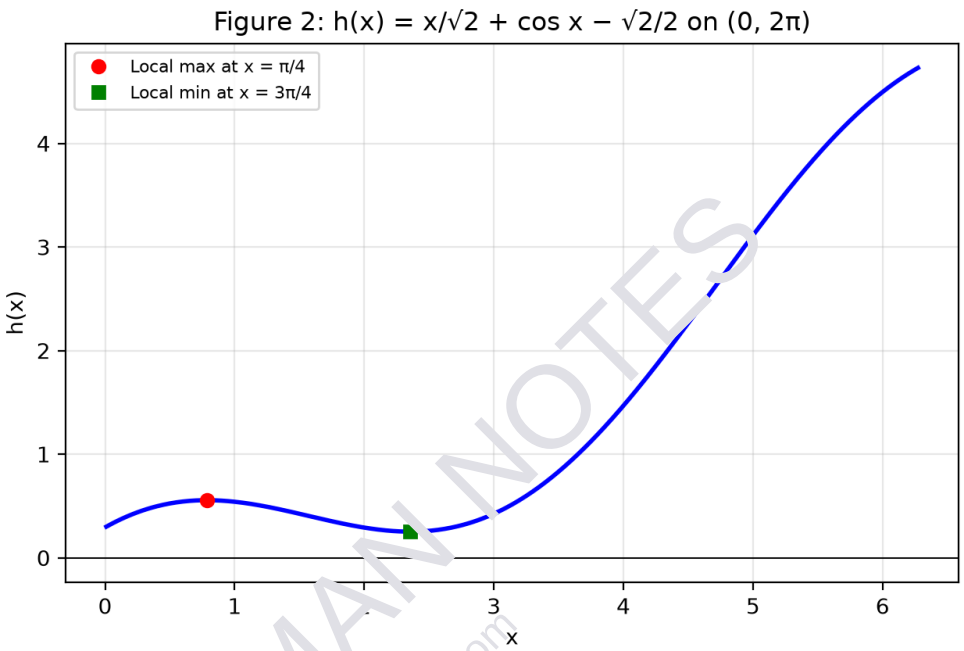
$(3\pi/4, 2\pi)$, $\sin x < 1/\sqrt{2}$ again (indeed $\sin x \le 0$ on $(\pi, 2\pi)$), so $h' > 0$ and h is increasing. The sign pattern +, −, + settles the classification:

Table 1: First derivative test for $h(x) = x/\sqrt{2} + \cos x - \sqrt{2}/2$

Interval / point	Sign of h'	Behaviour	Conclusion
$(0, \pi/4)$	+	Increasing	—
$x = \pi/4$	0	—	Local maximum, $h(\pi/4) = \pi\sqrt{2}/8 \approx 0.5554$
$(\pi/4, 3\pi/4)$	−	Decreasing	—
$x = 3\pi/4$	0	—	Local minimum, $h(3\pi/4) = \sqrt{2}(3\pi/8 - 1) \approx 0.2519$
$(3\pi/4, 2\pi)$	+	Increasing	—

The extreme values are computed exactly. At $x = \pi/4$: $h(\pi/4) = (\pi/4)/\sqrt{2} + \cos(\pi/4) - \sqrt{2}/2 = \pi/(4\sqrt{2}) + \sqrt{2}/2 - \sqrt{2}/2 = \pi/(4\sqrt{2}) = \pi\sqrt{2}/8 \approx 0.5554$, the two $\sqrt{2}/2$ terms cancelling exactly — an elegant feature of the problem's design. At $x = 3\pi/4$: $h(3\pi/4) = (3\pi/4)/\sqrt{2} + \cos(3\pi/4) - \sqrt{2}/2 = 3\pi/(4\sqrt{2}) - \sqrt{2}/2 - \sqrt{2}/2 = 3\pi\sqrt{2}/8 - \sqrt{2} = \sqrt{2}(3\pi/8 - 1) \approx 1.6661 - 1.4142 \approx 0.2519$. Figure 2 displays the curve with both extrema marked; notice how gentle the local minimum is compared with the subsequent steady rise driven by the linear term $x/\sqrt{2}$, which eventually dominates the bounded cosine term. Because the question asks for local extrema in the open interval $(0, 2\pi)$, the endpoints are not candidates; for the record, h rises to about $h(2\pi) \approx 4.729$ by the right end, far above both local extreme values, without contradicting their local character.

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The sign pattern of h' is +, −, + across x = π/4 and x = 3π/4, giving a local maximum at π/4 and a local minimum at 3π/4.

Conclusion

In part (a), expanding each term separately gives the fractional-power Maclaurin expansion f(x) = 1 − √2 x^{1/2} − x + x²/6 + (√2/3)x^{5/2} − x³/90 + x⁴/2520 − (√2/90)x^{9/2} + ⋯, valid for small x > 0 and verified both symbolically and numerically. In part (b), h′(x) = 1/√2 − sin x vanishes at x = π/4 and x = 3π/4 in (0, 2π), and the first derivative test (signs +, −, +) shows a local maximum at x = π/4 with value π√2/8 ≈ 0.5554 and a local minimum at x = 3π/4 with value √2(3π/8 − 1) ≈ 0.2519.