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Semester: Autumn 2026

Allama Iqbal Open University, Islamabad

Department of Mathematics

Course: Mathematics-II (1308)

Level: Intermediate

Semester: Autumn 2026

Assignment No. 1 (Units 1–5)

Total Marks: 100

Pass Marks: 40

SOLVED ASSIGNMENT GUIDE — For Study & Reference

Note: This guide is for study and reference only. Students must write their assignments in their own words; copying verbatim is plagiarism under AIOU policy. Booklet defects found in the printed assignment are flagged openly inside the relevant answers, with the printed form quoted exactly.

Question No. 1 [20]: (a) A jet is flying on a circular track of radius 176 meters at the rate of 570 km per hour. Through what angle will it turn in 2 seconds? (b) If θ = 15π/7, find values of all trigonometric ratios.

Introduction

This question brings together the two basic ways of measuring angles that run through the whole of trigonometry. Part (a) is an application of the relation between arc length, radius and the angle at the centre, $s = r\theta$, where θ must be measured in radians; it also requires a careful conversion of a speed given in kilometres per hour into metres per second, because the radius is given in metres and the time in seconds. Part (b) uses the opposite skill: instead of converting a length into an angle, we are given an angle in radian measure that is larger than one complete revolution, and

we must first reduce it to the equivalent angle inside one revolution before any trigonometric ratio can be evaluated. Both parts therefore test the same underlying idea, namely that radian measure is the natural measure of angle, with one full turn equal to 2π radians, and that every trigonometric ratio depends only on the position of the terminal arm of the angle, not on how many complete turns were made in reaching it.

Part (a): Angle Turned by the Jet in 2 Seconds

The jet moves on a circular track of radius r = 176 m. Its speed is 570 km per hour. Before anything else, the speed must be expressed in metres per second so that it is consistent with the radius in metres and the time in seconds. Since 1 km = 1,000 m and 1 hour = 3,600 s, we multiply by 1,000/3,600, that is, by 5/18:

$$v = 570 \times (5/18) = 158.33 \text{ m/s (approximately).}$$

In 2 seconds the jet therefore covers an arc length of $s = v \times t = 158.33 \times 2 = 316.67$ m along the circular track. The angle turned at the centre of the circle is obtained from the defining relation of radian measure, $\theta = s/r$, which is not an empirical formula but the very definition of a radian: the angle at the centre is the ratio of the arc it cuts off to the radius. Substituting the values gives:

$$\theta = s/r = 316.67/176 = 1.7992 \text{ radians (approximately).}$$

Because angles in everyday aviation and navigation are usually quoted in degrees, it is useful to convert this answer as well. Multiplying by $180/\pi$ gives $\theta = 1.7992 \times (180/\pi) = 103.09°$, approximately. Hence the jet turns through about 1.80 radians, that is about 103.1°, in 2 seconds. The result passes two quick sanity checks. First, the circumference of the track is $2\pi(176) = 1{,}105.8$ m, and the distance covered, 316.67 m, is about 0.286 of the circumference; 0.286 of a full turn of 360° is about 103°, which agrees with the computed value. Secondly, the angle is less than π radians (half a turn) but greater than π/2 (a quarter turn), exactly as the fraction 0.286 of a

revolution requires. Figure 1 shows the arc traced out. A common error in such problems is to insert the speed in kilometres per hour directly into s = rθ; that mixes units and produces a meaningless number. Unit conversion is therefore not a cosmetic first step but an essential part of the solution.

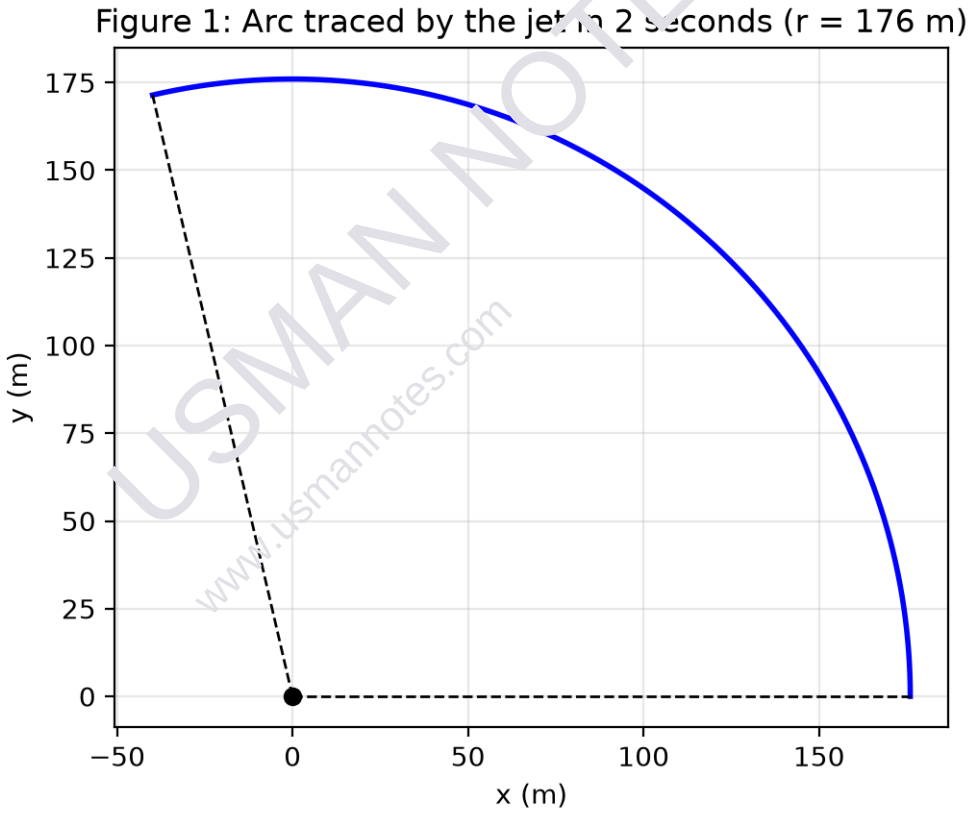


Figure 1: Arc traced by the jet in 2 seconds (r = 176 m)

The jet describes an arc of 316.67 m on a circle of radius 176 m, turning through θ = s/r ≈ 1.80 rad at the centre.

Part (b): All Trigonometric Ratios for θ = 15π/7

The given angle is θ = 15π/7 radians. Since 15π/7 = 2π + π/7, the angle is one complete revolution (2π) plus the acute angle π/7. A complete revolution brings the terminal arm back to exactly the same position, so every trigonometric ratio of 15π/7 equals the corresponding ratio of π/7. The angle π/7 is approximately 0.4488 radians, or about 25.71°, and it lies in the first

quadrant, where all six ratios are positive. This reduction step, subtracting a whole multiple of 2π, is called finding the co-terminal angle, and it is the standard first move whenever an angle exceeds 2π or is negative.

The angle $\pi/7$ does not have ratios expressible in the familiar simple surd forms that $\pi/6$, $\pi/4$ and $\pi/3$ possess, so the exact values are properly left in terms of $\pi/7$, with decimal equivalents (computed and cross-checked numerically) for practical use. The three primary ratios are $\sin(15\pi/7) = \sin(\pi/7) \approx 0.4339$, $\cos(15\pi/7) = \cos(\pi/7) \approx 0.9010$, and $\tan(15\pi/7) = \tan(\pi/7) \approx 0.4816$. The remaining three ratios are the reciprocals: cosecant is the reciprocal of sine, secant of cosine, and cotangent of tangent. The complete set is summarised in the table below.

Table 1: Trigonometric ratios of $\theta = 15\pi/7$ ($= \pi/7$ after removing one full turn)

Ratio	Exact form	Value (approx.)
$\sin\theta$	$\sin(\pi/7)$	0.4339
$\cos\theta$	$\cos(\pi/7)$	0.9010
$\tan\theta$	$\tan(\pi/7)$	0.4816
$\operatorname{cosec}\theta$ ($\csc\theta$)	$1/\sin(\pi/7)$	2.3048
$\sec\theta$	$1/\cos(\pi/7)$	1.1099
$\cot\theta$	$1/\tan(\pi/7)$	2.0765

These values can be verified independently of any calculator table by using the Pythagorean identity: $\sin^2(\pi/7) + \cos^2(\pi/7) \approx 0.4339^2 + 0.9010^2 = 0.1883 + 0.8118 = 1.0001$, which equals 1 up to rounding error, as it must. The quotient identity is also satisfied: $\sin(\pi/7)/\cos(\pi/7) \approx 0.4339/0.9010 = 0.4816 = \tan(\pi/7)$. Such internal checks are worth performing in an examination as well, because they catch a mis-keyed or mis-copied ratio immediately. Note also the sign discussion: had the co-terminal angle fallen in another quadrant, some ratios would have been

negative according to the CAST rule; here the terminal arm rests in the first quadrant, so all six values are positive.

Conclusion

In part (a) the jet, travelling at 158.33 m/s, covers 316.67 m in 2 seconds and turns through $\theta \approx 1.80$ radians $\approx 103.1°$ at the centre of its 176 m circular track. In part (b), removing one complete revolution from $15\pi/7$ leaves $\pi/7$ in the first quadrant, giving $\sin\theta \approx 0.4339$, $\cos\theta \approx 0.9010$, $\tan\theta \approx 0.4816$, $\operatorname{cosec}\theta \approx 2.3048$, $\sec\theta \approx 1.1099$ and $\cot\theta \approx 2.0765$. Both parts rest on radian measure: the definition $\theta = s/r$ in the first, and the periodicity of the trigonometric functions with period 2π in the second.

Question No. 2 [20]: (a) If $\tan\alpha = -3/4$ $(\pi/2 < \alpha < \pi)$ and $\sin\beta = 5/13$, find the value of $\sec(\alpha + \beta)$. (b) Express $\cot^4\alpha$ in terms of multiple of α.

Introduction

Part (a) is a compound-angle problem. To evaluate $\sec(\alpha + \beta)$ we need $\cos(\alpha + \beta)$, and the addition formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ requires the sine and cosine of each angle separately. The data, however, give us only $\tan\alpha$ for the first angle and only $\sin\beta$ for the second, so the first task is to reconstruct the remaining ratios from the given one in each case, using a right-angled triangle (the 3–4–5 and 5–12–13 Pythagorean triples) together with the quadrant information to fix the signs. Part (b) asks for a power of a trigonometric function to be rewritten using multiple angles, the technique, based on the double-angle identities, that prepares the ground for integrating powers of sines and cosines later in the course.

Part (a): Finding $\sec(\alpha + \beta)$

For the angle α we are told $\tan\alpha = -3/4$ with $\pi/2 < \alpha < \pi$, so α lies in the second quadrant. In the second quadrant sine is positive while cosine and tangent are negative, which is consistent with

the negative tangent given. From |tan α| = 3/4 we build the 3–4–5 triangle: opposite side 3, adjacent side 4, hypotenuse 5. Applying the second-quadrant signs gives sin α = 3/5 and cos α = −4/5. These satisfy sin²α + cos²α = 9/25 + 16/25 = 1, as required.

For the angle β we are told only that sin β = 5/13. The 5–12–13 triangle then gives |cos β| = 12/13, but the booklet does not state the quadrant of β, and this omission matters: sine is positive in both the first and the second quadrant, so cos β = 12/13 if β is in the first quadrant and cos β = −12/13 if β is in the second. A transparency note on this defect is given at the end of this answer; here both cases are worked completely so that the answer is correct under either reading. Applying the addition formula cos(α + β) = cos α cos β − sin α sin β with sin α = 3/5, cos α = −4/5 and sin β = 5/13:

Case 1 (β in Quadrant I, cos β = 12/13): cos(α + β) = (−4/5)(12/13) − (3/5)(5/13) = (−48 − 15)/65 = −63/65.

Case 2 (β in Quadrant II, cos β = −12/13): cos(α + β) = (−4/5)(−12/13) − (3/5)(5/13) = (48 − 15)/65 = 33/65.

Since secant is the reciprocal of cosine, the required values are sec(α + β) = −65/63 ≈ −1.0317 in Case 1 and sec(α + β) = 65/33 ≈ 1.9697 in Case 2. The table below organises the reconstruction. In many textbook versions of this standard problem β is intended to be acute, and under that convention the headline answer is sec(α + β) = −65/63. A numerical check confirms the arithmetic: α = π − tan⁻¹(3/4) ≈ 2.4981 rad and, in Case 1, β = sin⁻¹(5/13) ≈ 0.3948 rad, so α + β ≈ 2.8929 rad, whose cosine is ≈ −0.9692 = −63/65, and whose secant is ≈ −1.0317. In Case 2, β ≈ π − 0.3948 = 2.7468 rad, α + β ≈ 5.2449 rad, and cos(5.2449) ≈ 0.5077 = 33/65, again agreeing.

Table 1: Reconstructing sin and cos from the given data

Angle	Given	Triangle	sin	cos

α (Quadrant II)	tan α = −3/4	3–4–5	3/5	−4/5
β (Quadrant I case)	sin β = 5/13	5–12–13	5/13	+12/13
β (Quadrant II case)	sin β = 5/13	5–12–13	5/13	−12/13

Part (b): Expressing cot⁴α in Terms of Multiples of α

We write $\cot^4\alpha = \cos^4\alpha/\sin^4\alpha$ and treat the numerator and denominator separately with the double-angle identities. From $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$ we have $\cos^2\alpha = (1 + \cos 2\alpha)/2$ and $\sin^2\alpha = (1 - \cos 2\alpha)/2$. Squaring once more and applying the same identities to $\cos^2 2\alpha$ and $\sin^2 2\alpha$, where $\cos^2 2\alpha = (1 + \cos 4\alpha)/2$ and $\sin^2 2\alpha = (1 - \cos 4\alpha)/2$, gives the standard power-reduction formulas:

$$\cos^4\alpha = (3 + 4\cos 2\alpha + \cos 4\alpha)/8, \quad \sin^4\alpha = (3 - 4\cos 2\alpha + \cos 4\alpha)/8.$$

For example, $\cos^4\alpha = [(1 + \cos 2\alpha)/2]^2 = (1 + 2\cos 2\alpha + \cos^2 2\alpha)/4 = (1 + 2\cos 2\alpha + (1 + \cos 4\alpha)/2)/4 = (3 + 4\cos 2\alpha + \cos 4\alpha)/8$, and the sine calculation is identical with the sign of $\cos 2\alpha$ reversed. Dividing the two results, the factor 1/8 cancels and we obtain $\cot^4\alpha$ expressed entirely through multiples of α:

$$\cot^4\alpha = (3 + 4\cos 2\alpha + \cos 4\alpha) / (3 - 4\cos 2\alpha + \cos 4\alpha).$$

An equivalent, more compact form follows directly from $\cot^2\alpha = \cos^2\alpha/\sin^2\alpha = (1 + \cos 2\alpha)/(1 - \cos 2\alpha)$, namely $\cot^4\alpha = [(1 + \cos 2\alpha)/(1 - \cos 2\alpha)]^2$, and expanding the squares and reducing $\cos^2 2\alpha$ once more reproduces the boxed result, which is a useful internal check. The result was also verified numerically at several test angles (for instance at $\alpha = 0.7$ rad both sides equal 1.9868, and at $\alpha = 0.3$ rad both equal 109.2136). One honest remark is in order: unlike $\cos^4\alpha$ or $\sin^4\alpha$ alone, $\cot^4\alpha$ cannot be written as a finite sum of cosines of multiples of α, because $\cot^4\alpha$

becomes infinite at $\alpha = 0, \pi, \ldots$ whereas any finite sum of cosines is bounded; a ratio of multiple-angle expressions, as given above, is therefore the correct and complete form of the answer. The formula is valid wherever $\sin \alpha \neq 0$.

Transparency Note and Conclusion

Transparency note (booklet defect): in part (a) the booklet specifies the quadrant of α ($\pi/2 < \alpha < \pi$) but gives no quadrant or range for β. Because $\sin \beta = 5/13 > 0$ leaves two possibilities for $\cos \beta$, the problem as printed has two correct answers, $-65/63$ (β acute, the usual textbook intention) and $65/33$ (β in the second quadrant); both are derived above rather than one being silently assumed. In conclusion: the addition formula, fed with ratios reconstructed from the 3–4–5 and 5–12–13 triangles with quadrant-correct signs, gives $\sec(\alpha + \beta) = -65/63$ for acute β (and $65/33$ otherwise), while repeated use of the double-angle identities reduces the fourth power to $\cot^4\alpha = (3 + 4\cos 2\alpha + \cos 4\alpha)/(3 - 4\cos 2\alpha + \cos 4\alpha)$.

Question No. 3 [20]: (a) Prove that $(1 + \tan\theta \tan 2\theta)(\sin^6\theta + \cos^6\theta) = (1 - 3\sin^2\theta \cos^2\theta) \sec 2\theta$. (b) What is the period of $\sec(3x/11)$?

Introduction

Part (a) asks for the proof of a trigonometric identity that looks complicated but is in fact assembled from two simple pieces, each of which can be simplified independently: the factor $(1 + \tan\theta \tan 2\theta)$, which collapses to $\sec 2\theta$, and the sum of sixth powers $(\sin^6\theta + \cos^6\theta)$, which collapses to $(1 - 3\sin^2\theta \cos^2\theta)$. Recognising this structure before starting is the key to a short, clean proof. Part (b) is a short question about the period of a trigonometric function whose argument has been scaled, and it tests the general rule that replacing x by kx divides the period by $|k|$. Both parts reward knowing a small stock of fundamental identities — the tangent double-

angle formula, the algebraic identity $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$, and $\sin^2\theta + \cos^2\theta = 1$ — and applying them in the right order.

Part (a): Proof of the Identity

We work on the left-hand side and simplify its two factors separately. First consider $S = \sin^6\theta + \cos^6\theta$. Writing $\sin^6\theta = (\sin^2\theta)^3$ and $\cos^6\theta = (\cos^2\theta)^3$, set $a = \sin^2\theta$ and $b = \cos^2\theta$. The sum-of-cubes identity $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$ then gives, since $a + b = \sin^2\theta + \cos^2\theta = 1$:

$$S = (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta\,(\sin^2\theta + \cos^2\theta) = 1 - 3\sin^2\theta\cos^2\theta.$$

This is exactly the bracket appearing on the right-hand side of the identity to be proved, so the sixth-power factor needs no further work. Next consider $T = 1 + \tan\theta\tan 2\theta$. Use the double-angle formula $\tan 2\theta = 2\tan\theta/(1 - \tan^2\theta)$. Writing $t = \tan\theta$ for brevity:

$$T = 1 + t \cdot [2t/(1 - t^2)] = (1 - t^2 + 2t^2)/(1 - t^2) = (1 + t^2)/(1 - t^2).$$

Now divide numerator and denominator by $(1 + t^2)$ and recall that $\cos 2\theta = \cos^2\theta - \sin^2\theta$, while $1 = \cos^2\theta + \sin^2\theta$; equivalently, $\cos 2\theta = (1 - \tan^2\theta)/(1 + \tan^2\theta)$, the standard tangent form of the double-angle formula. Hence $(1 + t^2)/(1 - t^2) = 1/\cos 2\theta = \sec 2\theta$, so $T = \sec 2\theta$. Multiplying the two simplified factors together gives $(1 + \tan\theta\tan 2\theta)(\sin^6\theta + \cos^6\theta) = \sec 2\theta \cdot (1 - 3\sin^2\theta\cos^2\theta) = (1 - 3\sin^2\theta\cos^2\theta)\sec 2\theta$, which is exactly the right-hand side. The identity is therefore proved. A numerical spot check confirms it: at $\theta = 0.4$ rad, the left-hand side evaluates to approximately 0.8814 and the right-hand side to approximately 0.8814 as well (agreement to four decimal places).

Two remarks about validity complete the proof. The steps require $\tan\theta$, $\tan 2\theta$ and $\sec 2\theta$ to exist, so the identity holds wherever $\cos\theta \neq 0$ and $\cos 2\theta \neq 0$; at points where $\cos 2\theta = 0$ both sides are undefined together, so no exception arises. It is also worth noticing why the identity is designed the way it is: the constructor began with the two elementary facts $S = 1 - 3\sin^2\theta\cos^2\theta$

and T = sec 2θ and multiplied them together to disguise them. Proving identities is largely the skill of seeing through such disguises by simplifying each factor, bracket or side down to its simplest form.

Part (b): Period of sec(3x/11)

The secant function sec x has fundamental period 2π, the same as cos x, since sec x = 1/cos x and the reciprocal does not change the period. For a function of the form f(kx), the graph is the graph of f stretched or compressed horizontally by the factor 1/k, and the period becomes (period of f)/|k|. Here k = 3/11, so the period is:

$$\text{Period} = 2\pi \div (3/11) = 2\pi \times (11/3) = 22\pi/3.$$

The answer 22π/3 ≈ 23.04 can be checked directly from the definition of a period: sec(3(x + 22π/3)/11) = sec(3x/11 + (3/11)(22π/3)) = sec(3x/11 + 2π) = sec(3x/11), and no smaller positive shift works, because a shift T would require (3/11)T to be a multiple of 2π, the smallest positive possibility being (3/11)T = 2π. The general principle is worth stating explicitly, since it is used repeatedly in graph sketching: multiplying the variable by a constant k > 1 compresses the graph and shortens the period, while a constant between 0 and 1, such as 3/11 here, stretches the graph and lengthens the period in the inverse ratio. This is also why, in part (a) style questions, functions like tan 2θ repeat twice as fast as tan θ, whose period π becomes π/2 for tan 2θ.

It is also worth recording an alternative way of seeing the period result, because it connects part (b) with the graph work elsewhere in this assignment. The graph of y = sec(3x/11) is the graph of y = sec x stretched horizontally by the factor 11/3: every feature of the curve — each branch, each vertical asymptote — is carried 11/3 times farther from the y-axis. The asymptotes of sec x stand at x = π/2 + nπ, so those of sec(3x/11) stand where 3x/11 = π/2 + nπ, that is at x = 11π/6 + 11nπ/3. The gap between consecutive asymptotes is therefore 11π/3, and since one full period of

secant contains exactly two asymptote gaps (the curve completes one full repetition between every second pair of asymptotes, passing through its maximum and minimum in between), the period is $2 \times 11\pi/3 = 22\pi/3$, confirming the formula-based answer from the geometry of the graph itself. This asymptote-spacing check is a useful habit: for tangent and cotangent, whose periods are π rather than 2π, the period equals a single asymptote gap, while for sine, cosine, secant and cosecant it equals two.

Conclusion

The identity in part (a) is proved by simplifying each factor separately: the sum of sixth powers reduces, via $a^3 + b^3$ with $\sin^2\theta + \cos^2\theta = 1$, to $(1 - 3\sin^2\theta \cos^2\theta)$, and $(1 + \tan\theta \tan 2\theta)$ reduces, via the double-angle formula for tangent, to $\sec 2\theta$; their product is the required right-hand side. In part (b), scaling the argument by 3/11 stretches the period of secant from 2π to $22\pi/3$. Both results were confirmed by direct numerical evaluation in addition to the symbolic proofs.

Question No. 4 [20]: (a) Draw the graph of $y = \cos(3x/2)$; $x \in [-\pi/2, \pi/2]$. (b) Prove that $\pi/4 - \sin^{-1}(1/\sqrt{5}) = \cot^{-1}3$.

Introduction

Part (a) combines two skills: understanding how the factor 3/2 inside the cosine changes the standard cosine curve, and restricting attention to a stated interval, here the closed interval $[-\pi/2, \pi/2]$. The reliable method is to build a table of values at the strategically chosen points where the argument $3x/2$ passes through the familiar angles $0, \pm\pi/2, \pm 3\pi/4$ and so on, and then to join the plotted points with a smooth cosine-shaped curve. Part (b) is a proof involving inverse trigonometric functions. The standard strategy for such proofs is to give the awkward inverse expression a name, convert it into an ordinary trigonometric ratio by means of a right-angled triangle, and then use an addition or subtraction formula to reach the target form.

Part (a): Graph of y = cos(3x/2) on [−π/2, π/2]

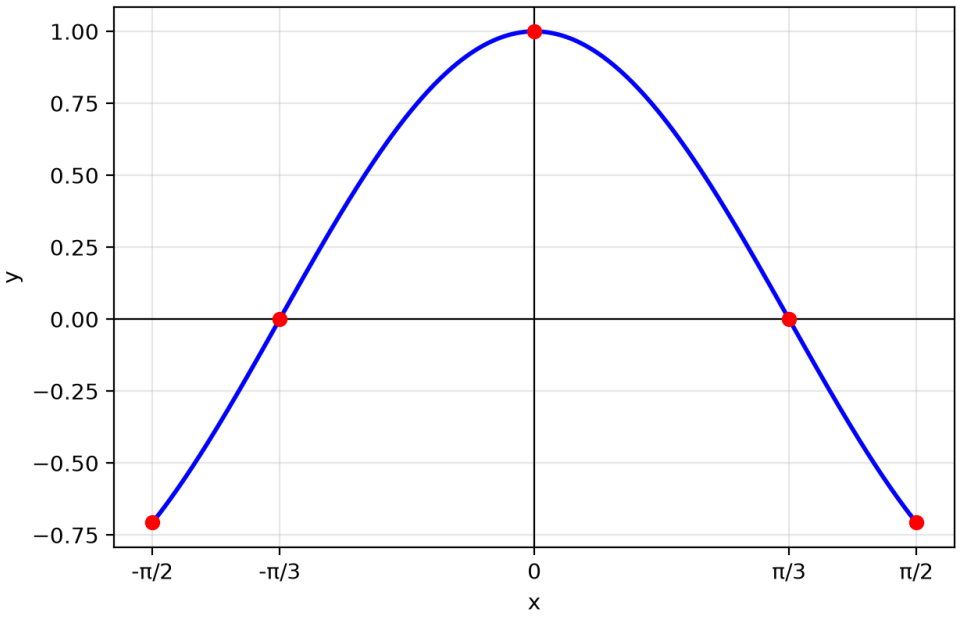
Let u = 3x/2, so y = cos u. As x runs from −π/2 to π/2, the argument u runs from −3π/4 to 3π/4. The cosine function is even, cos(−u) = cos u, so the graph is symmetric about the y-axis, and it attains its maximum value 1 at the centre x = 0 where u = 0. The function cos(3x/2) has period 2π/(3/2) = 4π/3, which is longer than the width π of our interval, so on this interval we see only the central arch of the cosine curve, falling from cos(−3π/4) = −√2/2 at the left endpoint up to 1 at x = 0 and back down to −√2/2 at the right endpoint. The curve crosses the x-axis where 3x/2 = ±π/2, that is at x = ±π/3. The table below gives the plotting values, and Figure 2 shows the resulting graph.

Table 1: Values of y = cos(3x/2) on [−π/2, π/2]

x	−π/2	−π/3	−π/6	0	π/6	π/3	π/2
3x/2	−3π/4	−π/2	−π/4	0	π/4	π/2	3π/4
y = cos(3x/2)	−0.7071	0	0.7071	1	0.7071	0	−0.7071

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Figure 2: y = cos(3x/2), x in [-π/2, π/2]



The graph is the central arch of the cosine curve: symmetric about the y-axis, maximum (0, 1), zeros at x = ±π/3, and endpoint values −√2/2 ≈ −0.7071 at x = ±π/2.

Reading the finished graph, the key features to label in an examination drawing are: the maximum point (0, 1); the two intercepts (−π/3, 0) and (π/3, 0); the endpoint values y = −√2/2 ≈ −0.707 at both ends; and the symmetry about the y-axis, which is a direct consequence of cosine being an even function. Because the interval is closed, both endpoints are included in the graph (shown as filled points in the figure). A student who knows the parent graph y = cos x can also obtain this graph by a horizontal stretch: replacing x by (3/2)x with the factor greater than 1 compresses horizontally, but since we only display the window [−π/2, π/2], what is visible is the portion of the compressed curve lying over that window, namely the single arch described above. Care is needed not to draw a full cosine wave: within this interval the function decreases monotonically on [0, π/2] and increases monotonically on [−π/2, 0], with no repetition.

The completed graph also answers several standard follow-up questions at a glance, and preparing for them is good examination practice. The domain is the given interval [−π/2, π/2] itself. The range is [−√2/2, 1], read from the lowest endpoint value and the maximum at the centre. The function is increasing on [−π/2, 0] and decreasing on [0, π/2], so its only turning point inside the interval is the maximum at x = 0. The equation cos(3x/2) = 1/2, for instance, would have exactly two solutions in the interval, symmetric about 0, at 3x/2 = ±π/3, i.e. x = ±2π/9 — a typical use of a sketch is to count and locate solutions of such equations before solving them algebraically. Sketching first and analysing second is the order in which this topic is meant to be used.

Part (b): Proof that $\pi/4 - \sin^{-1}(1/\sqrt{5}) = \cot^{-1}3$

Let $A = \sin^{-1}(1/\sqrt{5})$. By the definition of the inverse sine, A is the angle in [−π/2, π/2] with $\sin A = 1/\sqrt{5}$; since the value is positive, A is acute. Draw the corresponding right-angled triangle with

hypotenuse $\sqrt{5}$ and opposite side 1; the adjacent side is $\sqrt{(5-1)} = 2$. Therefore $\cos A = 2/\sqrt{5}$ and $\tan A = 1/2$. Now apply the tangent subtraction formula to $\pi/4 - A$, using $\tan(\pi/4) = 1$:

$$\tan(\pi/4 - A) = [\tan(\pi/4) - \tan A] / [1 + \tan(\pi/4)\tan A] = (1 - 1/2)/(1 + 1/2) = (1/2)/(3/2) = 1/3.$$

Hence $\pi/4 - A = \tan^{-1}(1/3)$, provided the angle $\pi/4 - A$ lies in the principal range $(-\pi/2, \pi/2)$ of $\tan^{-1}$, which it does: $A = \sin^{-1}(1/\sqrt{5}) \approx 0.4636$ rad $\approx 26.57°$, so $\pi/4 - A \approx 0.3218$ rad $\approx 18.43°$, comfortably inside the range, and the step of taking $\tan^{-1}$ of both sides is legitimate. Finally, $\tan^{-1}(1/3)$ and $\cot^{-1}3$ name the same angle, because if an angle has tangent 1/3 then its cotangent is the reciprocal, 3; formally $\cot^{-1}3 = \tan^{-1}(1/3)$ for positive values. Substituting back $A = \sin^{-1}(1/\sqrt{5})$ gives $\pi/4 - \sin^{-1}(1/\sqrt{5}) = \cot^{-1}3$, which is what was to be proved. The decimal values confirm the equality: $\pi/4 - 0.4636 = 0.3218$ rad, and $\cot^{-1}3 = \tan^{-1}(1/3) = 0.3218$ rad.

The method generalises, and it is worth recording the pattern for revision. To prove an identity containing inverse functions: (1) name the inverse terms as angles; (2) draw the triangle for each and read off all needed ratios; (3) apply the appropriate addition or subtraction formula to the combination; and (4) check that the combined angle lies in the principal range of the inverse function appearing in the conclusion, since inverse trigonometric functions are only single-valued on their principal ranges. Step (4) is the one most often omitted, and omitting it can turn a false statement into an apparently proved one when angles fall outside the principal ranges.

Conclusion

The graph of $y = \cos(3x/2)$ on $[-\pi/2, \pi/2]$ is a single symmetric cosine arch with maximum 1 at $x = 0$, zeros at $x = \pm\pi/3$, and value $-\sqrt{2}/2$ at both endpoints, as shown in the table and figure. The inverse-function identity follows by setting $A = \sin^{-1}(1/\sqrt{5})$, reading $\tan A = 1/2$ from the 1–2–$\sqrt{5}$ triangle, applying the tangent subtraction formula to get $\tan(\pi/4 - A) = 1/3$, and converting $\tan^{-1}(1/3)$ into $\cot^{-1}3$, with the principal-range check confirming every step.

Question No. 5 [20]: (a) Solve for "y" of cos²y + sin y = 1. (b) Show that (cos 23 + sin 23)/(cos 23 − sin 23) = tan 68.

Introduction

Part (a) asks us to solve a trigonometric equation, which means finding the general solution — all angles y, not merely those between 0° and 360° — satisfying cos²y + sin y = 1. Equations containing two different functions of the same angle are almost always attacked by first using a Pythagorean identity to express everything in terms of one function; here cos²y = 1 − sin²y converts the equation into a quadratic in sin y, which factorises neatly. Part (b) is a verification involving the specific degree measures 23° and 68°, whose sum is 90° and whose difference structure (68 = 45 + 23) points directly at the tangent addition formula. A note on the printing of part (b) is included below, because the booklet prints the numbers 23 and 68 without degree symbols.

Part (a): Solving cos²y + sin y = 1

Replace cos²y by 1 − sin²y, using sin²y + cos²y = 1. The equation becomes (1 − sin²y) + sin y = 1, and the 1 on each side cancels, leaving −sin²y + sin y = 0, or equivalently sin y (1 − sin y) = 0 after multiplying by −1 and factorising. A product is zero only when one of its factors is zero, so there are exactly two families of solutions:

Family 1: $\sin y = 0 \Rightarrow y = n\pi,\ n \in \mathbb{Z}$. Family 2: $\sin y = 1 \Rightarrow y = \pi/2 + 2n\pi,\ n \in \mathbb{Z}$.

In degree measure these are y = 180°n and y = 90° + 360°n respectively, for any integer n (positive, negative or zero). Both families genuinely satisfy the original equation, which is easily checked: if sin y = 0 then cos²y = 1 and cos²y + sin y = 1 + 0 = 1; if sin y = 1 then cos²y = 0 and cos²y + sin y = 0 + 1 = 1. Within one revolution, $0 \le y < 2\pi$, the solutions are y = 0, y = π/2 and y = π, a complete list that illustrates how the two general families specialise. It is instructive to see

why no other solutions exist: the substitution reduced the equation exactly to a quadratic in the single variable s = sin y, namely $s^2 - s = 0$, and a quadratic has at most two roots; since $|s| \leq 1$ automatically, neither root needs to be rejected as out of range — a rejection step that must never be skipped in equations where a root such as sin y = 2 can appear formally.

Graphically, the equation says that the curves $Y = \cos^2 y + \sin y$ and Y = 1 meet; since $\cos^2 y + \sin y = 1 - \sin^2 y + \sin y$ is a downward parabola in the variable sin y with values between 0 and 5/4, the level 1 is reached exactly at sin y = 0 and sin y = 1, matching the algebraic answer. Note also the nearby maximum: at sin y = 1/2 the left side equals 5/4, so the equation $\cos^2 y + \sin y = c$ has solutions only for c in a restricted range — a reminder that writing down the general solution, with the integer parameter n, is an essential part of 'solving' a trigonometric equation, since the functions involved are periodic and every solution repeats indefinitely.

Part (b): Showing that (cos 23° + sin 23°)/(cos 23° − sin 23°) = tan 68°

Transparency note: the booklet prints this part with bare numbers, '(cos 23 + sin 23)/(cos 23 − sin 23) = tan 68', without degree symbols. The identity is a degree identity — it rests on 68 = 45 + 23 — so the angles are read as 23° and 68° throughout, the standard convention in this course for such numerical identities; read as radians the statement would not hold and would have no reason to involve the number 68. With that reading, divide both the numerator and the denominator of the left-hand side by cos 23°, which is nonzero. Using sin 23°/cos 23° = tan 23°, the fraction becomes:

$$(\cos 23° + \sin 23°)/(\cos 23° - \sin 23°) = (1 + \tan 23°)/(1 - \tan 23°).$$

Now write 1 as tan 45° and apply the tangent addition formula $\tan(A + B) = (\tan A + \tan B)/(1 - \tan A \tan B)$ with A = 45° and B = 23°:

$$(\tan 45° + \tan 23°)/(1 - \tan 45° \tan 23°) = \tan(45° + 23°) = \tan 68°.$$

This is exactly the right-hand side, so the identity is shown. A numerical check removes any remaining doubt: cos 23° ≈ 0.9205 and sin 23° ≈ 0.3907, so the left-hand side is (0.9205 + 0.3907)/(0.9205 − 0.3907) = 1.3112/0.5298 ≈ 2.4751, while tan 68° ≈ 2.4751, in agreement to four decimal places. The same trick — dividing through by a cosine to convert an expression into tangents, then recognising an addition formula — is one of the most useful in elementary trigonometry; it also underlies the formula for tan(π/4 + θ) = (1 + tan θ)/(1 − tan θ), of which the present identity is the special case θ = 23°.

Conclusion

For part (a), converting cos²y into 1 − sin²y reduces the equation to sin y (1 − sin y) = 0, giving the general solutions y = nπ and y = π/2 + 2nπ (n any integer); in [0, 2π) these are 0, π/2 and π, each verified by substitution. For part (b), reading the bare numbers as degree measures (as the booklet's convention requires, and flagged in the note above), dividing numerator and denominator by cos 23° and applying the tangent addition formula with 45° + 23° = 68° proves that (cos 23° + sin 23°)/(cos 23° − sin 23°) = tan 68°, confirmed numerically as ≈ 2.4751 on both sides.

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