ALLAMA IQBAL OPEN UNIVERSITY

USMAN NOTES
www.usmannotes.com

Name:

Registration No.:

Course Code: 1307

Program: Intermediate

Semester: Autumn 2026

ALLAMA IQBAL OPEN UNIVERSITY, ISLAMABAD

Course: Mathematics-1 (1307)

Level: Intermediate | Department of Mathematics

Semester: Autumn 2026

Assignment No. 02 (Units. 6–9)

Total Marks: 100 | Pass Marks: 40

SOLVED ASSIGNMENT GUIDE — For Study & Reference

Note for students: This guide is for study and reference only. Read each solution, understand the method, and then write the assignment in your own words and your own hand. Copying any guide word-for-word is against the University's plagiarism policy, which is printed on the first page of your assignment booklet.

Booklet note: the cover schedule of this booklet lists submission dates for up to four assignments under the 6-credit-hour column, but the booklet itself prints only Assignment No. 01 (Units 1–5) and Assignment No. 02 (Units 6–9). This guide solves exactly the two assignments printed. A small number of printed questions contain internal inconsistencies — Assignment 1 Question 4(b) and Assignment 2 Question 4(a) — and these are flagged openly inside the relevant answers, with the question solved exactly as printed.

Question No. 1 [20]

Question No. 1: (a) Check the consistency of the system of linear equations given below: $2x_1 + x_2 - 2x_3 = 3$; $x_1 - x_2 - x_3 = 11$; $x_1 + x_2 + 3x_3 = 12$. (b) Show that the determinant $|2+\rho, 3, 4; 2, 3+\rho, 4; 2, 3, 4+\rho| = \rho^2(9+\rho)$.

Introduction

A system of linear equations is called consistent if it has at least one solution, and inconsistent if it has none. For three equations in three unknowns there are three possibilities: a unique solution (the three planes meet in one point), infinitely many solutions (the planes share a line or coincide), or no solution. The cleanest test uses determinants or, equivalently, ranks: if the determinant of the coefficient matrix is not zero, the system is certainly consistent with exactly one solution. Part (a) applies that test and then, for completeness, actually finds the solution. Part (b) is a determinant identity in the parameter p, proved by the standard row and column operations that leave a determinant's value unchanged or change it in controlled ways.

Part (a) — Consistency of the system

Write the system as A x = b with coefficient matrix A = [[2, 1, −2], [1, −1, −1], [1, 1, 3]] and constant column b = (3, 11, 12). Expanding det(A) along the first row: 2·[(−1)(3) − (−1)(1)] − 1·[(1)(3) − (−1)(1)] + (−2)·[(1)(1) − (−1)(1)] = 2(−3 + 1) − 1(3 + 1) − 2(1 + 1) = 2(−2) − 4 − 4 = −12. Since det(A) = −12 ≠ 0, the matrix A has rank 3, the augmented matrix also has rank 3, and rank(A) = rank(augmented) = number of unknowns. The system is therefore consistent, and in fact has a unique solution.

Finding the solution by elimination (or Cramer's rule, which the non-zero determinant licenses) gives

$$x_1 = 97/12 \approx 8.083, \quad x_2 = -19/3 \approx -6.333, \quad x_3 = 41/12 \approx 3.417.$$

Verification, equation by equation. First: 2(97/12) + (−19/3) − 2(41/12) = 194/12 − 76/12 − 82/12 = 36/12 = 3 ✓. Second: 97/12 − (−19/3) − 41/12 = 97/12 + 76/12 − 41/12 = 132/12 = 11 ✓. Third: 97/12 + (−19/3) + 3(41/12) = 97/12 − 76/12 + 123/12 = 144/12 = 12 ✓. All three equations are satisfied exactly, which is the concrete meaning of consistency here. For contrast, had det(A) been 0 we would have had to compare ranks of A and of the augmented matrix —

equal ranks would mean infinitely many solutions, unequal ranks inconsistency — but with determinant −12 none of that machinery is needed.

Part (b) — The determinant identity

Let D denote the determinant with rows (2 + ρ, 3, 4), (2, 3 + ρ, 4), (2, 3, 4 + ρ). The trick is to notice that if we add the second and third columns to the first column — the operation $C_1 \to C_1 + C_2 + C_3$, which does not change a determinant's value — every entry of the new first column becomes the same: the first row gives (2 + ρ) + 3 + 4 = 9 + ρ, the second gives 2 + (3 + ρ) + 4 = 9 + ρ, and the third gives 2 + 3 + (4 + ρ) = 9 + ρ. Factor 9 + ρ out of that column:

$$D = (9 + \rho) \cdot | 1, 3, 4; 1, 3 + \rho, 4; 1, 3, 4 + \rho |.$$

Now subtract the first row from the second and from the third — the operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, which again leave the value unchanged. The first column becomes (1, 0, 0); the second row becomes (0, ρ, 0) because 3 + ρ − 3 = ρ and 4 − 4 = 0; the third row becomes (0, 0, ρ) because 3 − 3 = 0 and 4 + ρ − 4 = ρ. The determinant is now upper triangular:

$$D = (9 + \rho) \cdot | 1, 3, 4; 0, \rho, 0; 0, 0, \rho | = (9 + \rho) \cdot (1 \cdot \rho \cdot \rho) = \rho^2(9 + \rho),$$

since the value of a triangular determinant is the product of its diagonal entries. This is exactly what was to be shown. Two independent spot-checks protect against slips: putting ρ = 0 makes two rows of the original determinant proportional in the pattern (2,3,4) shifted — directly, the original becomes |2,3,4; 2,3,4; 2,3,4|-type rows all equal to (2, 3, 4) in pattern... more precisely at ρ = 0 every row is (2, 3, 4) except the diagonal increments vanish, so all three rows are identical and the determinant is 0, matching $\rho^2(9 + \rho) = 0$. Putting ρ = 1 and expanding numerically gives $1^2 \cdot 10 = 10$, and direct expansion of |3,3,4; 2,4,4; 2,3,5| = 3(20 − 12) − 3(10 − 8) + 4(6 − 8) = 24 − 6 − 8 = 10 ✓.

Conclusion

The system's coefficient determinant is $-12 \neq 0$, so the system is consistent with the unique solution (97/12, −19/3, 41/12), verified in all three equations. The determinant identity follows from two value-preserving operations — adding all columns into the first to expose the factor $9 + \rho$, then clearing below the diagonal to expose $\rho \cdot \rho$ — giving $\rho^2(9 + \rho)$, confirmed at $\rho = 0$ and $\rho = 1$. In both parts the winning strategy is identical: transform first, compute second.

Geometric picture and the rank criterion

It may help to see part (a) geometrically. Each equation in three unknowns represents a plane in three-dimensional space. A unique solution means the three planes pass through one common point and no other — here the point (97/12, −19/3, 41/12) — like the corner of a room where two walls meet the floor. Inconsistency would mean the planes form a triangular prism arrangement with no point common to all three, and infinitely many solutions would mean all three planes share a whole line. Algebraically all three situations are read off from two numbers, the rank of the coefficient matrix and the rank of the augmented matrix: equal ranks mean consistency, and rank 3 with three unknowns means uniqueness. The determinant test used in the solution is simply the fastest route to those ranks when, as here, the determinant does not vanish. This picture also explains why 'check the consistency' questions sometimes ask only for consistency and not for the solution: consistency is a property of the ranks alone, settled before any elimination is carried out.

Question No. 2 [20]

Question No. 2: (a) Find the sum of the first 75 terms of the sequence whose (n)th term is given by $x_n = (-7)(2/7)^n$. (b) If the 5th term and 11th term of an H.P are 3/7 and 9/13 respectively, find the sequence.

Introduction

This question pairs the two workhorse progressions of the unit. Part (a)'s sequence is geometric: each term is a fixed multiple — the common ratio — of the one before, and its partial sums have a closed formula. Part (b) concerns a harmonic progression (H.P.), whose terms are the reciprocals of an arithmetic progression (A.P.); every H.P. problem is solved by taking reciprocals, doing ordinary A.P. work, and turning the answer back over. Watching which of the two patterns a sequence follows — constant ratio, or reciprocals with constant difference — is the whole skill being tested.

Part (a) — Sum of the first 75 terms

With $x_n = (-7)(2/7)^n$, write out the start of the sequence using the standard convention that the first term corresponds to n = 1: $x_1 = (-7)(2/7) = -2$, $x_2 = (-7)(2/7)^2 = -4/7$, $x_3 = -8/49$, and so on. Each term is 2/7 of its predecessor, so this is a geometric progression with first term a = −2 and common ratio r = 2/7. The sum of the first N terms of a G.P. is S_N = a(1 − r^N)/(1 − r). With N = 75:

$$S_{75} = -2 \cdot [\, 1 - (2/7)^{75} \,] / (1 - 2/7) = -2 \cdot [\, 1 - (2/7)^{75} \,] / (5/7) = -(14/5) \cdot [\, 1 - (2/7)^{75} \,].$$

That is the exact answer. Its size is easy to state: $(2/7)^{75}$ is about 10^−41, utterly negligible, so $S_{75} \approx -14/5 = -2.8$ to any accuracy anyone could demand. This is also the sum to infinity of the progression, S_∞ = a/(1 − r) = −2/(5/7) = −14/5, which exists because |r| = 2/7 < 1. The partial sums therefore race to −2.8: already $S_1 = -2$, $S_2 \approx -2.571$, $S_3 \approx -2.735$, $S_5 \approx -2.791$. A frequent error is to take the first term as −7 (the value the formula gives at n = 0); the phrase 'first 75 terms of the sequence whose nth term is x_n' counts from n = 1, and the check $x_1 = -2$, $x_2/x_1 = 2/7$ confirms the identification of a and r before any formula is used.

Part (b) — Recovering the H.P. from its 5th and 11th terms

Let the corresponding A.P. — the reciprocals of our H.P. terms — have first term A and common difference D. The 5th term of the H.P. is 3/7, so the 5th term of the A.P. is 7/3; the 11th term of the H.P. is 9/13, so the 11th term of the A.P. is 13/9. Hence A + 4D = 7/3 and A + 10D = 13/9. Subtracting the first equation from the second: 6D = 13/9 − 7/3 = 13/9 − 21/9 = −8/9, so D = −4/27. Back-substituting: A = 7/3 − 4D = 7/3 + 16/27 = 63/27 + 16/27 = 79/27.

The A.P. is therefore 79/27, 75/27, 71/27, 67/27, 63/27, … with general term $A_n = 79/27 - (4/27)(n - 1) = (83 - 4n)/27$. Taking reciprocals term by term, the required harmonic progression is

27/79, 9/25, 27/71, 27/67, 3/7, 27/59, 27/55, 9/17, … , with general term 27/(83 − 4n).

Verification against both given data points: at n = 5 the general term gives 27/(83 − 20) = 27/63 = 3/7 ✓, and at n = 11 it gives 27/(83 − 44) = 27/39 = 9/13 ✓. Notice the character of the answer: because the underlying A.P. has a negative common difference, its terms shrink, so the H.P. terms themselves grow (0.342, 0.360, 0.380, 0.403, 0.429, …), as the figure shows. Students often find a growing H.P. surprising, but nothing in the definition forbids it — an H.P. is simply whatever the reciprocals of an A.P. happen to do. The one fatal error to avoid is working with 3/7 and 9/13 as if they were A.P. terms directly; the reciprocal step at the start is not optional decoration, it is the method.

USMAN NOTES www.usmannotes.com

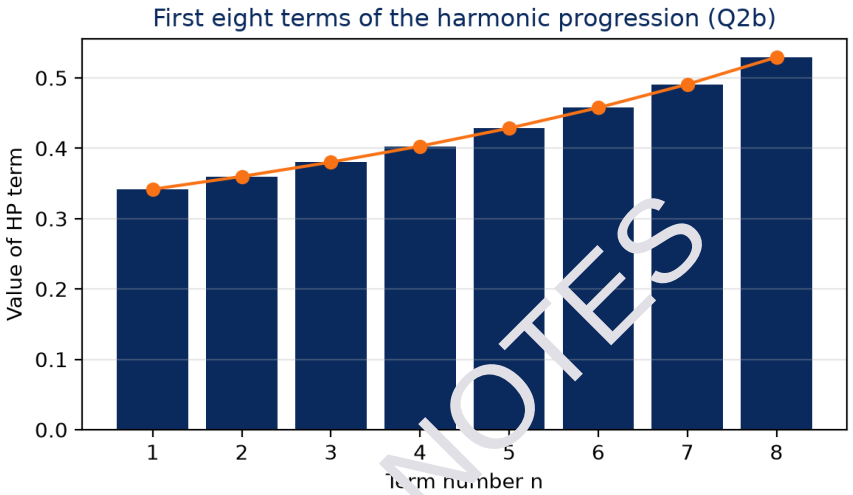


Figure 3: The first eight terms of the harmonic progression of part (b), increasing because the underlying A.P. decreases.

USMAN NOTES www.usmannotes.com

Conclusion

For part (a), the sequence is geometric with a = −2 and r = 2/7, and the sum of its first 75 terms is $-(14/5)[1-(2/7)^{75}] \approx -2.8$. For part (b), passing to reciprocals gives an A.P. with A = 79/27 and D = −4/27, so the H.P. has general term 27/(83 − 4n) and begins 27/79, 9/25, 27/71, 27/67, 3/7, …, reproducing both given terms exactly.

Two quick consistency checks on the answers

Small cases are the cheapest auditors of general formulas, and both parts survive them. For part (a), summing the first two terms directly gives $-2 - 4/7 = -18/7 \approx -2.571$, while the formula gives $-(14/5)[1-(2/7)^2] = -(14/5)(45/49) = -18/7$ ✓; summing ten terms by the formula gives $-(14/5)[1-(2/7)^{10}] \approx -2.8000$ already to four decimal places, showing how quickly a ratio as small as 2/7 exhausts the series. For part (b), extend the H.P. two terms beyond those displayed using the general term 27/(83 − 4n): the 9th term is 27/47 ≈ 0.574 and the 10th is 27/43 ≈ 0.628, continuing the steady increase. Notice that the terms will keep growing as long as the A.P. denominators 83 − 4n stay positive, that is for $n \leq 20$; beyond that the reciprocal A.P. terms turn

negative and the H.P. terms become negative too. The given data only asks for 'the sequence', so displaying the first several terms with the general term, as in the solution, is a complete answer within the range where the terms behave as the first eleven do.

One final identification point ties the two halves together: a progression is recognised by what stays constant. In part (a) the ratio of successive terms stays constant at 2/7, which is the signature of a G.P.; in part (b) nothing about the H.P. terms themselves is constant — neither their differences nor their ratios — but the differences of their reciprocals stay constant at −4/27, which is the signature of an H.P. Testing both signatures on any new sequence takes only three terms and a few seconds, and it prevents the most expensive mistake in this unit, applying the sum formula of one progression to a sequence that belongs to the other.

Question No. 3 [20]

Question No. 3: (a) Mr. John invests $1865 at 7% interest compounded annually. Find the amount he receives after 3 years. (b) How many new words can be formed from the word "OSTEOPOROSIS"?

Introduction

The two halves of this question come from different corners of the course — financial arithmetic and combinatorics — but they share a moral: both are one-formula problems in which all the real work lies in identifying the quantities correctly before substituting. In part (a) the formula is the compound amount A = P(1 + r)^t, and the traps are the meaning of 'compounded annually' and the difference between the amount received and the interest earned. In part (b) the formula counts permutations of letters with repetitions, and the trap is that the word OSTEOPOROSIS does not have twelve different letters.

Part (a) — Compound amount on $1865 at 7% for 3 years

When interest is compounded annually, the interest earned in each year is itself added to the principal, so the following year's interest is computed on a larger amount. If P is the principal, r the annual rate as a decimal and t the number of years, the amount after t years is A = P(1 + r)^t. Here P = 1865 dollars, r = 7% = 0.07 and t = 3, so

$$A = 1865 \times (1.07)^3 = 1865 \times 1.225043 = \$2284.71 \text{ (to the nearest cent).}$$

The year-by-year table shows the compounding at work and provides a check on the power:

Year	Amount at start of year ($)	Interest at 7% ($)	Amount at end of year ($)
1	1865.00	130.55	1995.55
2	1995.55	139.69	2135.24
3	2135.24	149.47	2284.71

Reading down the interest column shows why compound interest outruns simple interest: the interest itself grows each year (130.55, then 139.69, then 149.47) because each year it is charged on a bigger balance. For comparison, simple interest would have paid a flat $130.55 every year, totalling $391.65 and an amount of $2256.65; compounding pays $419.71 of interest instead — the extra $28.06 is 'interest on interest'. So Mr. John receives approximately $2284.71 after 3 years, of which $419.71 is interest. (More precisely the amount is $2284.705…, and quoting $2284.71 assumes the usual rounding to cents.)

Part (b) — Arrangements of the letters of OSTEOPOROSIS

Count the letters first, carefully, because everything else follows from the count. OSTEOPOROSIS has 12 letters. Writing them out — O, S, T, E, O, P, O, R, O, S, I, S — and tallying gives: O occurs 4 times, S occurs 3 times, and T, E, P, R and I occur once each. Check: 4 + 3 + 1 + 1 + 1 + 1 + 1 = 12 ✓.

Letter	O	S	T	E	P	R	I	Total
Occurrences	4	3	1	1	1	1	1	12

If all 12 letters were different there would be 12! arrangements. But swapping two identical letters — two O's, say — does not produce a new word, and the 12! count has counted each distinct word once for every internal shuffling of its repeated letters: 4! shufflings of the O's and 3! shufflings of the S's. Dividing out this overcounting, the number of distinct words (arrangements) that can be formed is

$$12! / (4! \times 3!) = 479001600 / (24 \times 6) = 479001600 / 144 = 3{,}326{,}400.$$

A word on the phrasing 'new words'. The figure 3,326,400 counts all distinct arrangements of the twelve letters, one of which is the original word OSTEOPOROSIS itself. If 'new' is pressed literally to exclude the original arrangement, the number of new words is 3,326,400 − 1 = 3,326,399. In examination practice the standard expected answer is the full count 3,326,400, and the subtraction remark is best included as a clarifying note exactly as done here, so that the answer is right under either reading. Also note that 'words' in such problems means arrangements of letters, whether or not they appear in a dictionary — checking 3.3 million strings against a dictionary is neither intended nor possible.

Conclusion

Compounded annually at 7%, $1865 grows through $1995.55 and $2135.24 to $2284.71 in three years, earning $419.71 of interest. The letters of OSTEOPOROSIS (O four times, S three times, five letters once) can be arranged in 12!/(4!·3!) = 3,326,400 distinct words — 3,326,399 of them new if the original word itself is excluded. In both parts the formula is the easy step; the marks are earned in the set-up: the correct rate-time reading, and the correct letter tally.

Why the division principle in part (b) is correct — a miniature example

Students often accept the formula of part (b) on authority; a three-letter miniature shows why it must be true. Consider arranging the letters A, A, B. Treating the A's as distinct (A_1, A_2, B) gives 3! = 6 arrangements: A_1A_2B, A_2A_1B, A_1BA_2, A_2BA_1, BA_1A_2, BA_2A_1. But erasing the subscripts, these collapse in pairs to only three genuinely different words: AAB, ABA, BAA. Each true word was counted exactly 2! = 2 times, once for each ordering of the identical A's, so the true count is 3!/2! = 3. Exactly the same collapse operates in OSTEOPOROSIS, with 4! orderings of the O's and 3! orderings of the S's inflating the count, and dividing by 4! × 3! deflates it back to the number of genuinely different words, 3,326,400. The same miniature also warns against the commonest error in such problems: dividing only by one of the repetition factorials, or worse, subtracting the repeats instead of dividing by them. Repetition inflates a count multiplicatively, so only division can undo it. A similar sanity habit applies to part (a): before trusting $(1.07)^3$, estimate — 7% of $1265 is about $130 a year, three years of simple interest is about $390, and compounding must give somewhat more than the resulting $2255; the computed $2284.71 sits exactly where the estimate says it should.

A last remark on presentation: money answers should always be given to two decimal places with the currency unit attached, and counting answers should be given as exact whole numbers, never as decimals. Writing '$2284.71' and '3,326,400 words' rather than bare numbers is not pedantry; it records what the number measures, and in both parts of this question the unit is part of what the examiner is checking. Where an exact value and an approximation both exist — as with the compound amount $2284.705… — give the exact computation first and the rounded value second, so that rounding is seen to be a final presentational step rather than a source of error accumulated along the way.

Question No. 4 [20]

Question No. 4: (a) A box contains 75 pins marked 7, 8, 9 ----- 82. A pin is randomly picked from the box. Find the probability that the pin marked is "7 or a multiple of 5". (b) Expand (1 + 5/(2x))^n and find its general term. Find the condition under which this expansion is valid.

Introduction

Part (a) is a classical-probability problem: when every outcome is equally likely, probability is the count of favourable outcomes divided by the count of all outcomes, so the whole task is careful counting — and this printed question contains a counting trap of the booklet's own making, which we flag openly below. Part (b) asks for the binomial expansion of (1 + 5/(2x))^n, its general term, and the condition of validity, which depends on whether n is a positive integer or a general index; a complete answer treats both cases.

Part (a) — Probability of "7 or a multiple of 5"

Booklet inconsistency, stated before any computation: the pins are said to be 75 in number, but they are also said to be marked 7, 8, 9, …, 82, and the integers from 7 to 82 inclusive number 82 − 7 + 1 = 76, not 75. Both statements cannot be true at once. The marks determine what can actually be drawn, so the correct sample space is the 76 marks 7 through 82, and the main answer below uses 76; the result under the booklet's printed count of 75 is also given, so the answer is usable whichever figure your tutor works with.

Now count the favourable outcomes. The event is 'the mark is 7, or the mark is a multiple of 5'. The multiples of 5 between 7 and 82 inclusive are 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80 — that is 15 marks. The mark 7 is not itself a multiple of 5, so the two parts of the 'or' do not overlap and no outcome is double-counted; the favourable count is 15 + 1 = 16. (This disjointness check matters: had the singled-out mark been, say, 10, it would already be among the multiples and the favourable count would have stayed 15.) Hence

$$P(7 \text{ or a multiple of } 5) = 16/76 = 4/19 \approx 0.2105, \quad \text{i.e. about } 21.05\%.$$

Under the booklet's printed total of 75 pins the same favourable count gives $16/75 \approx 0.2133$ (about 21.33%). The difference between 4/19 and 16/75 is small but real, and it comes entirely from the booklet's internal inconsistency, not from any ambiguity in the method: favourable outcomes divided by total outcomes, with equal likelihood guaranteed by the words 'randomly picked'. One more habit worth forming: always sanity check a probability answer — it must lie between 0 and 1, and a favourable set of 16 out of roughly 76 outcomes should indeed give a little over one fifth, as 4/19 does.

Part (b) — Expansion of $(1 + 5/(2x))^n$, general term and validity

The binomial theorem states that for a positive integer n, $(1 + u)^n = 1 + n\,u + n(n-1)/2! \cdot u^2 + n(n-1)(n-2)/3! \cdot u^3 + \ldots$, a finite sum of n + 1 terms, where the coefficient of u^r is the binomial coefficient $C(n, r) = n!/[r!(n-r)!]$. Here $u = 5/(2x)$. Substituting, the expansion is

$$(1 + 5/(2x))^n = 1 + n\cdot(5/(2x)) + [n(n-1)/2!]\cdot(5/(2x))^2 + [n(n-1)(n-2)/3!]\cdot(5/(2x))^3 + \ldots$$

$$= 1 + 5n/(2x) + 25n(n-1)/(8x^2) + 125n(n-1)(n-2)/(48x^3) + \ldots$$

The general term — the (r + 1)th term, with r running from 0 to n — is obtained by inserting u = 5/(2x) into T_{r+1} = C(n, r) u^r:

$$T_{r+1} = C(n, r) \cdot (5/(2x))^r = C(n, r) \cdot 5^r / (2^r x^r).$$

Check it against the displayed terms: r = 0 gives 1; r = 1 gives $n \cdot 5/(2x)$ ✓; r = 2 gives $n(n-1)/2 \cdot 25/(4x^2) = 25n(n-1)/(8x^2)$ ✓; r = 3 gives $n(n-1)(n-2)/6 \cdot 125/(8x^3) = 125n(n-1)(n-2)/(48x^3)$ ✓. Note the powers of x are in the denominator: the expansion is a series in descending powers of x (ascending powers of 1/x), and it has no meaning at x = 0, where the original expression itself is undefined.

Validity now splits into the two cases the wording invites. Case 1: n is a positive integer. The series terminates after n + 1 terms, an identity between polynomials in 1/x, and it is valid for every x ≠ 0 with no further restriction. Case 2: n is not a positive integer (a fraction or a negative number). The binomial series then does not terminate; the coefficients are formed with the generalised product n(n − 1)(n − 2)…(n − r + 1)/r!, and the infinite series converges to (1 + 5/(2x))^n only when the added term is less than 1 in absolute value, |5/(2x)| < 1, that is |2x| > 5, or

|x| > 5/2.

This convergence condition is what the question's phrase 'the condition under which this expansion is valid' most likely targets, since for integer n the question would be empty. Summarise it as: the expansion is an exact finite identity for positive integral n and all x ≠ 0; as an infinite binomial series for general n it is valid precisely when |x| > 5/2 (with x ≠ 0 understood).

Conclusion

For part (a) the favourable marks are 7 together with the 15 multiples of 5, giving probability 16/76 = 4/19 ≈ 0.2105 on the marks actually printed, 7 to 82 (16/75 ≈ 0.2133 if the booklet's stated 75 is used — the discrepancy is the booklet's, and it is flagged rather than hidden). For part (b), (1 + 5/(2x))^n expands with general term T_{r+1} = C(n, r)·5^r/(2^r x^r), valid for all x ≠ 0 when n is a positive integer, and for |x| > 5/2 when the expansion is used as an infinite series for general n.

Question No. 5 [20]

Question No. 5: (a) Find the sum of the series: 1 + 5/7 + (5·7)/(7·14) + (5·7·9)/(7·14·21) + … (b) Use mathematical induction to prove that $1^3 + 3^3 + 5^3 + \dots + (2x - 1)^3 = x^2(2x^2 - 1)$.

Introduction

Part (a) gives a series whose terms are written in a deliberately un-simplified dress; the way to sum such a series is to recognise which standard expansion it is. The reliable method is comparison: write down the candidate standard series, compute its first few terms, and check them against the given ones one by one — recognition by verification, not by guesswork. Part (b) is a proof by mathematical induction, the formal ladder argument: verify the first rung, then show that standing on any rung lets you step to the next. Note that the booklet prints the counting variable as x (most books use n); x here is a positive integer counting how many odd cubes are added, and we treat it exactly as printed.

Part (a) — Summing the series

First tidy the given terms. The third term is (5·7)/(7·14) = 35/98 = 5/14, and the fourth is (5·7·9)/(7·14·21) = 315/2058 = 15/98. So the series is 1 + 5/7 + 5/14 + 15/98 + …, and successive ratios are (5/7), (1/2), (3/7), … — the pattern of ratios (5/7, then (7/2)/…) is a fingerprint of a binomial series with index −5/2. Test the candidate (1 − z)^(−5/2) with z = 2/7. Its expansion, from the general binomial theorem (1 − z)^(−5/2) = 1 + (5/2)z + [(5/2)(7/2)/2!]z² + [(5/2)(7/2)(9/2)/3!]z³ + …, gives, term by term at z = 2/7:

Term	From (1 − 2/7)^(−5/2)	Value	Given series (simplified)
1st	1	1	1
2nd	(5/2)(2/7)	5/7	5/7
3rd	(5/2)(7/2)/2! × (2/7)²	5/14	(5·7)/(7·14) = 5/14
4th	(5/2)(7/2)(9/2)/3! × (2/7)³	15/98	(5·7·9)/(7·14·21) = 15/98

All four printed terms match exactly, and the match is structural, not coincidental: in the given series the numerator of each new term picks up the next odd factor (5, 7, 9, …) while the denominator picks up the next multiple of 7 (7, 14, 21, …), which is precisely what the coefficients (5/2)(7/2)(9/2)… and the powers (2/7)^k produce after simplification. The series is therefore the binomial expansion of (1 − 2/7)^(−5/2), and since |2/7| < 1 the infinite series converges to that value. Hence the sum is

$$S = (1 - 2/7)^{(-5/2)} = (5/7)^{(-5/2)} = (7/5)^{(5/2)} = 49\sqrt{35} / 125 \approx 2.3191.$$

The last exact form comes from $(7/5)^{(5/2)} = (7/5)^2 \cdot \sqrt{(7/5)} = (49/25)\cdot(\sqrt{35}/5) = 49\sqrt{35}/125$. A numerical check by partial sums shows the convergence clearly: $S_1 = 1$, $S_2 \approx 1.7143$, $S_3 \approx 2.0714$, $S_4 \approx 2.2245$, and the sums continue to creep upward toward 2.3191, never overshooting — all terms are positive and the increments shrink, exactly as the figure displays. The corresponding general term, for reference, is the coefficient pattern above: the (k + 1)th term is [(5·7·9⋯(2k + 3)] / [7·14·21⋯(7k)] with the empty product read as 1 for k = 0.

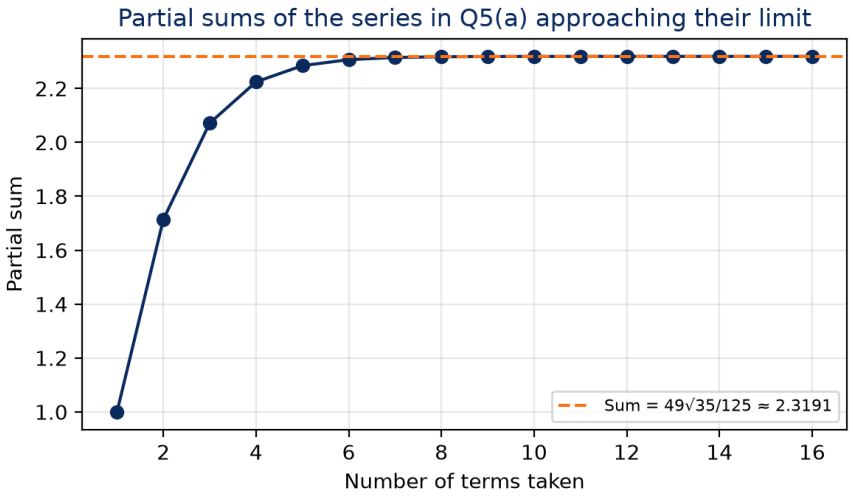


Figure 4: Partial sums of the series in part (a) approaching the limit $49\sqrt{35}/125 \approx 2.3191$.

Part (b) — Induction proof for the sum of the first x odd cubes

Let P(x) be the statement $1^3 + 3^3 + 5^3 + \dots + (2x - 1)^3 = x^2(2x^2 - 1)$, for a positive integer x. Step 1 — base case x = 1: the left side is just $1^3 = 1$, and the right side is $1^2(2 \cdot 1^2 - 1) = 1 \times 1 = 1$. They agree, so P(1) is true. (Checking one further case costs nothing and often catches a misread formula: x = 2 gives 1 + 27 = 28 on the left and 4(8 − 1) = 28 on the right ✓.)

Step 2 — inductive step: assume P(k) is true for some positive integer k, that is, $1^3 + 3^3 + \dots + (2k - 1)^3 = k^2(2k^2 - 1)$. The next odd number after $2k - 1$ is $2k + 1 = 2(k + 1) - 1$, so the sum up to $(2(k + 1) - 1)^3$ equals the assumed sum plus the one new term $(2k + 1)^3$:

$$\text{LHS for } x = k + 1 = k^2(2k^2 - 1) + (2k + 1)^3.$$

Expand both pieces: $k^2(2k^2 - 1) = 2k^4 - k^2$, and $(2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1$. Adding, the total is $2k^4 + 8k^3 + 11k^2 + 6k + 1$. Now compute what P(k + 1) requires the answer to be: $(k + 1)^2(2(k + 1)^2 - 1) = (k^2 + 2k + 1)(2k^2 + 4k + 1)$. Multiplying out: $k^2 \cdot (2k^2 + 4k + 1) = 2k^4 + 4k^3 + k^2$; $2k \cdot (2k^2 + 4k + 1) = 4k^3 + 8k^2 + 2k$; $1 \cdot (2k^2 + 4k + 1) = 2k^2 + 4k + 1$; and the sum is $2k^4 + 8k^3 + 11k^2 + 6k + 1$ — identical to the expression obtained from the inductive assumption, term for term. Hence the formula holds for k + 1 whenever it holds for k, which is exactly P(k) ⇒ P(k + 1).

Step 3 — conclusion: P(1) is true, and truth at any stage forces truth at the next; by the principle of mathematical induction P(x) is true for every positive integer x. That is,

$$1^3 + 3^3 + 5^3 + \dots + (2x - 1)^3 = x^2(2x^2 - 1) \quad \text{for all positive integers } x, \quad \blacksquare$$

Two features of this proof deserve a sentence each, because examiners probe them. First, the inductive step must use the assumption — we substituted $k^2(2k^2 - 1)$ for the old sum rather than re-deriving it; an 'induction' that never uses its hypothesis is not an induction. Second, induction proves the formula for the counting variable x = 1, 2, 3, … only; the identity is a statement about

sums of finitely many cubes and has no meaning for non-integer x, which is why the booklet's use of the letter x (rather than n) must be read as a positive integer throughout.

Conclusion

The series of part (a) is the binomial expansion of (1 − 2/7)^(−5/2) in disguise, verified term by term, and its sum is (7/5)^(5/2) = 49√35/125 ≈ 2.3191. The identity of part (b) is proved by induction: it holds at x = 1, and the step from k to k + 1 is confirmed by the exact agreement of the two expansions $2k^4 + 8k^3 + 11k^2 + 6k + 1$. Recognition-by-verification in the first part and hypothesis-driven algebra in the second are the two techniques this question exists to teach.

USMAN NOTES www.usmannotes.com