ALLAMA IQBAL OPEN UNIVERSITY

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Name:

Registration No.:

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Program: Intermediate

Semester: Autumn 2026

ALLAMA IQBAL OPEN UNIVERSITY, ISLAMABAD

Course: Mathematics-1 (1307)

Level: Intermediate | Department of Mathematics

Semester: Autumn 2026

Assignment No. 01 (Units. 1–5)

Total Marks: 100 | Pass Marks: 40

SOLVED ASSIGNMENT GUIDE — For Study & Reference

Note for students: This guide is for study and reference only. Read each solution, understand the method, and then write the assignment in your own words and your own hand. Copying any guide word-for-word is against the University's plagiarism policy, which is printed on the first page of your assignment booklet.

Booklet note: the cover schedule of this booklet lists submission dates for up to four assignments under the 6-credit-hour column, but the booklet itself prints only Assignment No. 01 (Units 1–5) and Assignment No. 02 (Units 6–9). This guide solves exactly the two assignments printed. A small number of printed questions contain internal inconsistencies — Assignment 1 Question 4(b) and Assignment 2 Question 4(a) — and these are flagged openly inside the relevant answers, with the question solved exactly as printed.

Question No. 1 [20]

Question No. 1: (a) Prove that √5 is an irrational number. (b) Check the set {1, √2, 1/√2} for the closure property with respect to addition and multiplication.

Introduction

This question brings together two of the most basic ideas of the first unit of the course: what kind of number √5 is, and what it means for a set of numbers to be closed under an operation. Part (a)

is proved by the classical method of contradiction, the same method Euclid used for √2 more than two thousand years ago. Part (b) is settled by direct computation: we form every possible sum and every possible product of pairs of elements of the given set, using tables, and we simply look at whether the results stay inside the set. Both parts reward care rather than cleverness — a single counter-example is enough to destroy a closure claim, while the irrationality proof needs each logical step to be watertight.

Part (a) — √5 is irrational

Recall the definitions first. A rational number is a number that can be written as p/q where p and q are integers, q ≠ 0, and the fraction may always be reduced to its lowest terms, that is, so that p and q have no common factor except 1 (we say p and q are co-prime). An irrational number is a real number that is not rational. To prove that √5 is irrational we assume the opposite and derive a contradiction.

Suppose, if possible, that √5 is rational. Then there exist integers p and q, with q ≠ 0, in lowest terms, such that $\sqrt{5} = p/q$. Squaring both sides gives $5 = p^2/q^2$, that is,

$$p^2 = 5q^2.$$

This equation says that p^2 is a multiple of 5, because it equals 5 times the integer q^2. Now a key fact about the prime number 5 is needed: if a prime divides the square of an integer, it divides the integer itself. (If 5 did not divide p, then p would leave remainder 1, 2, 3 or 4 on division by 5, and p^2 would leave remainder 1 or 4, never 0.) Hence 5 divides p, so we may write $p = 5k$ for some integer k. Substituting in $p^2 = 5q^2$ gives $25k^2 = 5q^2$, and dividing by 5,

$$q^2 = 5k^2.$$

But this says exactly the same thing about q: q^2 is a multiple of 5, so by the same fact, 5 divides q as well. We have therefore shown that 5 is a common factor of both p and q. That contradicts our

opening choice of p/q in lowest terms, where the only common factor allowed is 1. The contradiction shows that the original supposition is impossible; √5 cannot be written as a fraction of integers at all. Hence √5 is an irrational number, which completes the proof.

Two remarks are worth adding for full marks. First, the argument uses in an essential way that 5 is prime; the same proof works for √p for every prime p, but it collapses for, say, √4 — correctly, because √4 = 2 is rational. Second, irrationality agrees with the decimal picture: √5 = 2.2360679… continues forever without any repeating block, whereas every rational number has a decimal that either terminates or eventually repeats. The figure below shows √5 geometrically, as the hypotenuse of a right-angled triangle with sides 1 and 2, by Pythagoras' theorem: √(1² + 2²) = √5.

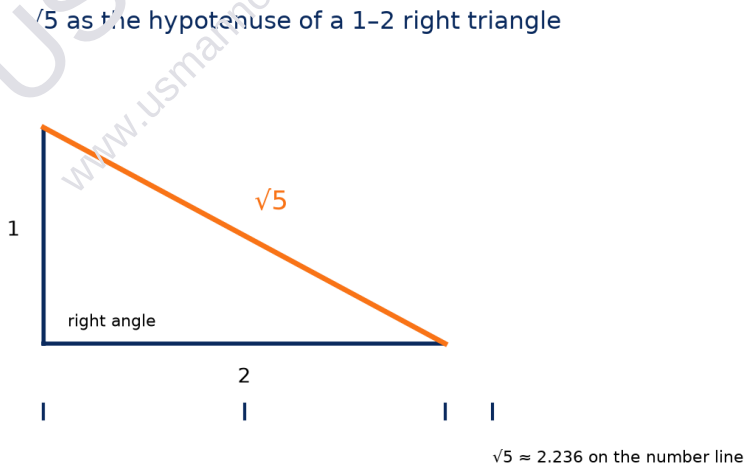


Figure 1: √5 constructed as the hypotenuse of a right-angled triangle with legs 1 and 2.

Part (b) — Closure of the set {1, √2, 1/√2}

A set S is said to be closed under an operation if, whenever the operation is applied to any two elements of S (the same element may be used twice), the result is again an element of S. To test closure honestly we must check all pairs, or produce one failing pair — a counter-example — which is enough to declare the set not closed. Let S = {1, √2, 1/√2}. Note for the computations

that $1/\sqrt{2} = \sqrt{2}/2 \approx 0.7071$ and $\sqrt{2} \approx 1.4142$. The complete addition and multiplication tables are as follows.

+	1	√2	1/√2
1	2	1 + √2	1 + 1/√2
√2	1 + √2	2√2	√2 + 1/√2
1/√2	1 + 1/√2	√2 + 1/√2	√2

×	1	√2	1/√2
1	1	√2	1/√2
√2	√2	2	1
1/√2	1/√2	1	1/2

Read the addition table first. The simplest entry already fails: $1 + 1 = 2$, and 2 is not an element of S, whose members are approximately 1, 1.4142 and 0.7071. In fact almost every sum fails — $1 + \sqrt{2} \approx 2.4142$, $2\sqrt{2} \approx 2.8284$ and $\sqrt{2} + 1/\sqrt{2} \approx 2.1213$ are all outside S — and only the single entry $1/\sqrt{2} + 1/\sqrt{2} = \sqrt{2}$ happens to land back in the set. Since closure demands that every pair works, one failing pair is decisive: S is NOT closed with respect to addition.

The multiplication table fails too. The cleanest counter-example is $\sqrt{2} \times \sqrt{2} = 2$, which is not in S. A second one is $(1/\sqrt{2}) \times (1/\sqrt{2}) = 1/2$, also not in S. It is true that several products do return to the set — anything multiplied by 1 stays where it was, and $\sqrt{2} \times 1/\sqrt{2} = 1$ — but closure is an all-pairs property, not a some-pairs property, so these successes cannot rescue the claim. Hence S is NOT closed with respect to multiplication either.

Conclusion

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Part (a) is settled by contradiction: assuming $\sqrt{5} = p/q$ in lowest terms forces 5 to divide both p and q, which is impossible, so $\sqrt{5}$ is irrational. Part (b) is settled by counter-examples displayed in the two tables: $1 + 1 = 2 \notin S$ destroys closure under addition, and $\sqrt{2} \times \sqrt{2} = 2 \notin S$ destroys closure under multiplication. The moral for the rest of the course is that number systems such as the rationals and the reals earn their importance precisely because they are closed under these operations, while small finite sets like S almost never are.

A further remark on why these two results matter

It is worth pausing on how the two halves of this question support each other. The irrationality of numbers such as $\sqrt{2}$ and $\sqrt{5}$ is precisely why closure questions like part (b) are interesting: the set $\{1, \sqrt{2}, 1/\sqrt{2}\}$ mixes a rational number with two irrational ones, and its failure to be closed shows that such ad-hoc collections behave nothing like the familiar number systems. The rational numbers, by contrast, are closed under both addition and multiplication — the sum and product of any two fractions is again a fraction — and so are the real numbers. That closure is what makes ordinary algebra possible: we can add and multiply freely inside those systems without ever falling out of them. The given set fails at the very first hurdle, $1 + 1 = 2$, which is the simplest possible counter-example and the one to present first in an examination answer, before the multiplication counter-example $\sqrt{2} \times \sqrt{2} = 2$.

Question No. 2 [20]

Question No. 2: (a) Write the expression $(2 - 5i)/(1 + i)$ in the form $x + iy$. (b) If $z_1 = 3 - 2i$, $z_2 = 2 + i\sqrt{3}$, $z_3 = 1 - 9i$ then evaluate $| i z_1 / (z_2 + z_3) |$.

Introduction

Complex numbers extend the real numbers by adjoining the symbol i, where $i^2 = -1$. Every complex number can be written uniquely in the form $x + iy$ with x, y real; x is its real part and y

its imaginary part. Division of complex numbers is made routine by one device — multiplying numerator and denominator by the conjugate of the denominator — and the modulus $|z| = \sqrt{(x^2 + y^2)}$ measures the distance of z from the origin in the Argand diagram. Part (a) uses the conjugate device; part (b) uses the two most useful modulus rules, $|zw| = |z||w|$ and $|z/w| = |z|/|w|$, which save us from ever having to carry out the division itself.

Part (a) — Reducing (2 − 5i)/(1 + i) to the form x + iy

The conjugate of the denominator 1 + i is 1 − i. Multiplying the numerator and the denominator by 1 − i does not change the value of the fraction, but it turns the denominator into a real number, because $(1 + i)(1 - i) = 1 - i^2 = 1 + 1 = 2$. The numerator expands as $(2 - 5i)(1 - i) = 2 - 2i - 5i + 5i^2 = 2 - 7i - 5 = -3 - 7i$, where we have used $i^2 = -1$ in the last step. Therefore

$$(2 - 5i)/(1 + i) = (-3 - 7i)/2 = -3/2 - (7/2)i.$$

So the required form x + iy is obtained with x = −3/2 and y = −7/2. A quick check confirms the arithmetic: multiplying our answer back by the denominator, $(-3/2 - (7/2)i)(1 + i) = -3/2 - (3/2)i - (7/2)i - (7/2)i^2 = -3/2 - 5i + 7/2 = 2 - 5i$, which is exactly the original numerator. Students lose marks most often at two points in this standard routine: forgetting that $i^2 = -1$ turns $+5i^2$ into −5, and dividing only one term of the numerator by 2. Writing the check multiplication, as above, is cheap insurance against both errors.

Part (b) — Evaluating $|\, i z_1 / (z_2 + z_3) \,|$

By the quotient and product rules for modulus, the required modulus splits into pieces that need no division at all:

$$|\, i z_1 / (z_2 + z_3) \,| = |i| \cdot |z_1| / |z_2 + z_3|.$$

Compute each piece in turn. First, |i| = 1, since i = 0 + 1i and $\sqrt{(0^2 + 1^2)} = 1$; geometrically, multiplying by i rotates a point through a right angle about the origin, which never changes its

distance from the origin. Next, we may either use z_1 directly, $|z_1| = |3 - 2i| = \sqrt{(9 + 4)} = \sqrt{13}$, or first form the product $i\,z_1 = i(3 - 2i) = 3i - 2i^2 = 2 + 3i$, whose modulus is $\sqrt{(4 + 9)} = \sqrt{13}$ — the same value, as the rotation remark promised. Finally, the sum in the denominator is $z_2 + z_3 = (2 + i\sqrt{3}) + (1 - 9i) = 3 + (\sqrt{3} - 9)i$, because the real parts 2 and 1 add, and the imaginary coefficients $\sqrt{3}$ and -9 add. Its modulus is

$$|z_2 + z_3| = \sqrt{(\,3^2 + (\sqrt{3} - 9)^2\,)} = \sqrt{(\,9 + 3 - 18\sqrt{3} + 81\,)} = \sqrt{(93 - 18\sqrt{3})} \approx 7.855.$$

Putting the pieces together,

$$|\,i\,z_1 / (z_2 + z_3)\,| = \sqrt{13} / \sqrt{(93 - 18\sqrt{3})} \approx 3.606 / 7.855 \approx 0.4586.$$

If a rationalised single-fraction form is wanted, squaring and rationalising inside the root gives $13/(93 - 18\sqrt{3}) = (403 + 78\sqrt{3})/2559$, so the modulus also equals $\sqrt{[(403 + 78\sqrt{3})/2559]}$; the decimal value 0.4586 is unchanged. Notice that the answer is a positive real number less than 1: that is sensible, because the numerator $i\,z_1$ is a comparatively small vector (length about 3.61) while the denominator $z_2 + z_3$ is a long vector (length about 7.86), so their quotient must be short. The Argand diagram below plots all the numbers involved; reading lengths off such a diagram is a fast way to catch gross errors before they reach the answer line.

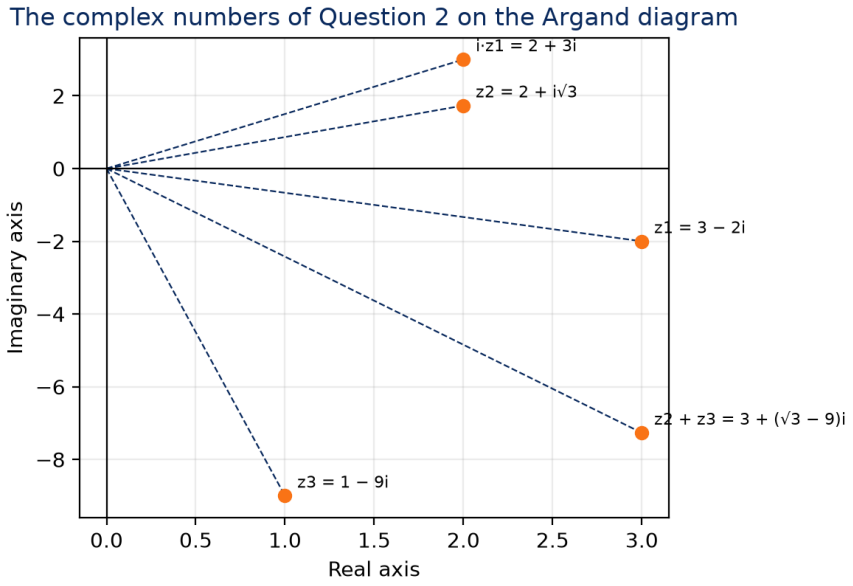


Figure 2: z_1, z_2, z_3, $i\,z_1$ and $z_2 + z_3$ on the Argand diagram; modulus is distance from the origin.

Conclusion

Rationalising with the conjugate gives $(2 - 5i)/(1 + i) = -3/2 - (7/2)i$, so $x = -3/2$ and $y = -7/2$, verified by multiplying back. For part (b), the product and quotient rules for modulus reduce the work to three easy lengths, and $|\, i\, z_1 / (z_2 + z_3)\,| = \sqrt{13}/\sqrt{(93 - 18\sqrt{3})} \approx 0.4586$. The general lesson — always split a modulus of products and quotients before attempting any division — applies to every modulus problem in this unit.

Direct-division cross-check for part (b)

As an independent check on the modulus rules, we can perform the division in part (b) outright and then take the modulus of the result. The quotient is $(2 + 3i) / [3 + (\sqrt{3} - 9)i]$. Rationalising with the conjugate $3 - (\sqrt{3} - 9)i = 3 + (9 - \sqrt{3})i$, the numerator becomes $(2 + 3i)(3 + (9 - \sqrt{3})i) = (-21 + 3\sqrt{3}) + (27 - 2\sqrt{3})i \approx -15.80 + 23.54i$, and the denominator becomes $3^2 + (9 - \sqrt{3})^2 = 93 - 18\sqrt{3} \approx 61.72$. The quotient is therefore approximately $-0.256 + 0.381i$, and its modulus is $\sqrt{(0.256^2 + 0.381^2)} \approx \sqrt{0.211} \approx 0.459$, matching the value 0.4586 obtained by the product-and-quotient rules, up to rounding. The agreement of the two routes — the long way and the short way — is the strongest evidence that no slip has been made, and performing the long way once, as here, is what justifies trusting the short way in the examination hall.

Question No. 3 [20]

Question No. 3: (a) Construct the truth table for the statement $p \rightarrow q \wedge (\sim p \rightarrow q) \vee p$. (b) Find the solution of the following equation, if any: 3^(2+x) − 3^(1−x) = 2.

Introduction

Part (a) belongs to mathematical logic, the machinery that underlies every proof in this course. A statement built from simple propositions p and q by connectives — negation ~, conjunction ∧,

disjunction ∨ and implication → — has its truth value completely determined by the truth values of p and q, and a truth table simply lists all four possible cases and works the value out column by column. Part (b) is an exponential equation; its solution turns on one idea, namely replacing the unknown expression 3^x by a single letter so that the equation becomes an ordinary quadratic, and then remembering that 3^x is always positive, which disposes of one of the two algebraic roots.

Part (a) — Truth table for p → q ∧ (~p → q) ∨ p

A note on grouping is needed before any table can be drawn, because the booklet prints the statement without brackets. We follow the standard precedence used throughout the course book: ~ binds most tightly, then ∧, then ∨, and the implication → is worked out last, governing the whole statement. Read this way the statement is p → [(q ∧ (~p → q)) ∨ p]: the part inside the brackets is built first, and the leading p → is applied at the end. The table below builds the statement exactly in that order, one connective at a time, which is also the safest examination technique — never try to jump straight to the final column.

p	q	~p	p → q	~p → q	q ∧ (~p → q)	[q ∧ (~p → q)] ∨ p	p → ([q ∧ (~p → q)] ∨ p)
T	T	F	T	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	T	T
F	F	T	T	F	F	F	T

Let us walk through the rows so that the table is not just copied but understood. The implication u → v is false in exactly one situation — u true and v false — and true in the other three; that

single rule generates both the p → q column and the ~p → q column. In row 2, p is true and q false, so p → q is false; but ~p is false, and an implication with false hypothesis (~p → q) is automatically true, so q ∧ (~p → q) = F ∧ T = F. In row 4 both p and q are false: ~p is true, so ~p → q is the one bad case T → F and is false, giving q ∧ (~p → q) = F and the bracket […] ∨ p = F ∨ F = F. The final column then reads: row 1, T → T = T; row 2, T → T = T; row 3, F → T = T; row 4, F → F = T.

The final column contains T in every row. A compound statement that is true for every assignment of truth values to its components is called a tautology, so the given statement is a tautology: it carries no information about p and q at all, because it can never be false. Tautologies of exactly this shape are what make proof by cases legitimate in mathematics — dividing an argument into 'p holds' and 'p fails' can never lead outside the truth. If the statement had come out false in every row it would have been a contradiction, and if mixed, a contingency; naming which of the three it is, is part of a complete answer.

Part (b) — Solving 3^(2+x) − 3^(1−x) = 2

Use the laws of exponents to pull the unknown into a single expression. Since $3^{(2+x)} = 3^2 \cdot 3^x = 9 \cdot 3^x$ and $3^{(1-x)} = 3^1 / 3^x = 3/3^x$, the equation becomes $9 \cdot 3^x - 3/3^x = 2$. Now substitute $t = 3^x$. Because a power of a positive number is always positive, we record the condition $t > 0$ immediately — it will be needed in a moment. The equation in t is $9t - 3/t = 2$, and multiplying through by t (which is safe, since $t \neq 0$) gives the quadratic

$$9t^2 - 2t - 3 = 0.$$

The quadratic formula gives $t = [2 \pm \sqrt{(4 + 108)}]/18 = [2 \pm \sqrt{112}]/18 = (1 \pm 2\sqrt{7})/9$. The two candidates are $t = (1 + 2\sqrt{7})/9 \approx 0.6991$ and $t = (1 - 2\sqrt{7})/9 \approx -0.4768$. The negative candidate must be rejected: t stands for 3^x, and $3^x > 0$ for every real x, so no real x can produce it. This

is exactly why we recorded t > 0 before solving — students who skip that step routinely submit two 'solutions' where only one exists. Keeping the positive root, 3^x = (1 + 2√7)/9, and taking logarithms to base 3,

$$x = \log_3 [\ (1 + 2\sqrt{7})/9\ ] \approx -0.326.$$

Verification by substitution: with x ≈ −0.326, 3^(2+x) = 3^1.674 ≈ 6.289 and 3^(1−x) = 3^1.326 ≈ 4.289, and 6.289 − 4.289 = 2.000, as required. (The exact check works the same way: 9t − 3/t with t = (1 + 2√7)/9 simplifies, using t's quadratic 9t² − 2t − 3 = 0 divided by t, to exactly 2.)

Conclusion

The truth table shows the printed statement — grouped by the standard precedence of connectives — to be a tautology, true in all four rows. The exponential equation has exactly one solution, x = log₃[(1 + 2√7)/9] ≈ −0.326; its quadratic in t = 3^x produces a second, negative root, which is extraneous because a real power of 3 can never be negative. Both parts illustrate the same habit: state the governing rule first (truth conditions of →; positivity of 3^x), then let the computation follow it.

A note on reading implications

One habit that the truth-table exercise is designed to build is suspicion of the everyday meaning of 'if … then'. In ordinary speech 'if p then q' often suggests that p causes q, or that q is false when p is false; in logic neither suggestion is present. The implication p → q makes exactly one promise — that the combination 'p true, q false' will not occur — and it is vacuously true whenever p is false, which is why the rows of the table with p false cause beginners the most discomfort. Keeping that single promise in mind generates the whole p → q column without memorisation. It also explains why the compound statement examined here can be a tautology

even though its ingredients look contingent: the final implication has, in every row, either a false hypothesis or a true conclusion available to it.

Question No. 4 [20]

Question No. 4: (a) Find the condition that one root of a $x^2 + b x + c = 0$ is the third-fifth of the difference of it from others. (b) A number consists of two digits whose product is 36 and on interchanging digits, the result exceeds the original number by 14. Find such numbers.

Introduction

Both parts concern quadratic equations and the relations between their roots and coefficients. For $a x^2 + b x + q = 0$ with roots α and β we shall use the standard results $\alpha + \beta = -b/a$ and $\alpha\beta = q/a$ (the constant term is printed as q on this booklet, where many books write c). Part (a) asks for a condition — an equation connecting a, b and q — that must hold when the roots stand in a stated relation. Part (b), taken exactly as printed, turns out to describe a number that cannot exist; showing that rigorously, and identifying the likely intended version, is the mathematically honest answer, and this guide does exactly that rather than inventing a number.

Part (a) — The condition on a, b and q

Interpretation, stated openly: 'the third-fifth' is the fraction 3/5, and 'the difference of it from the other [root]' is the other root minus this root. So the condition in words is that one root, call it α, equals 3/5 of $(\beta - \alpha)$, where β is the other root: $\alpha = (3/5)(\beta - \alpha)$. Multiplying out, $5\alpha = 3\beta - 3\alpha$, hence $8\alpha = 3\beta$, that is $\beta = 8\alpha/3$. The two roots therefore stand in the ratio 3 : 8. (If neither root is zero this is the only possibility; the degenerate case $\alpha = \beta = 0$ would force $b = q = 0$ and the 'equation' would collapse, so we set it aside.)

Now feed the ratio into the sum and product of roots. The sum is $\alpha + \beta = \alpha + 8\alpha/3 = 11\alpha/3$, and by the coefficient relation this equals $-b/a$, so $\alpha = -3b/(11a)$. The product is $\alpha\beta = \alpha \cdot (8\alpha/3) =$

$8\alpha^2/3$, and this equals q/a. Substituting $\alpha = -3b/(11a)$ gives $8/3 \cdot 9b^2/(121a^2) = 24b^2/(121a^2) = q/a$. Multiplying both sides by $121a^2$ clears the fractions and leaves the required condition in its cleanest form:

$$121\, a\, q = 24\, b^2.$$

There is a neat way to see why the numbers 121 and 24 appear, which is worth learning because it generalises. If two roots are in the ratio m : n, then $\alpha\beta/(\alpha + \beta)^2 = mn/(m + n)^2$. Here m : n = 3 : 8, so the ratio is 24/121. But $\alpha\beta/(\alpha + \beta)^2 = (q/a) / (b^2/a^2) = aq/b^2$. Setting $aq/b^2 = 24/121$ gives $121aq = 24b^2$ at once. As a numerical sanity check, take roots 3 and 8 themselves: the equation is $(x - 3)(x - 8) = x^2 - 11x + 24 = 0$, so $a = 1$, $b = -11$, $q = 24$, and indeed $121 \cdot 1 \cdot 24 = 2904 = 24 \cdot 121 = 24 \cdot (-11)^2$. The condition is satisfied, confirming the derivation.

Part (b) — A necessary note on the question as printed

Write the two-digit number as $10a + b$, where a is the tens digit and b the units digit. The first printed condition says $ab = 36$. Since a and b are single digits, the only possibilities are (a, b) = (4, 9), (6, 6) and (9, 4), giving the numbers 49, 66 and 94 respectively. Now apply the second printed condition. Interchanging the digits gives $10b + a$, and the gain on interchanging is $(10b + a) - (10a + b) = 9b - 9a = 9(b - a)$. For our three candidates the gains are $94 - 49 = 45$, $66 - 66 = 0$ and $49 - 94 = -45$. None of them is 14.

This failure is not an accident of the product 36; it could never have succeeded. The gain on reversing any two-digit number is $9(b - a)$, always a multiple of 9, whereas 14 is not a multiple of 9. So there is NO two-digit number — with digit-product 36 or with any other digit-product — whose reversal exceeds it by 14. The conclusion for the question exactly as printed is therefore: no such number exists; the two printed conditions are inconsistent.

Tens digit a	Units digit b	Number	Reversed	Gain on

			number	reversing
4	9	49	94	45
6	6	66	66	0
9	4	94	49	−45

Transparency note (read carefully): the '14' in the booklet is almost certainly a misprint for '45' — the classic textbook form of this problem — because with 45 everything fits perfectly: the number 49 has digit-product 4 × 9 = 36, and reversing it gives 94, which exceeds 49 by exactly 45. If your tutor confirms that 45 was intended, the answer to submit for that version is 49, found by solving ab = 36 with 9(b − a) = 45, i.e. b − a = 5, giving a = 4, b = 9. But that correction is a conjecture about a misprint, not something printed in the booklet, and this guide will not silently substitute it. In your own submission, present the inconsistency proof above — it is a complete, correct piece of mathematics for the question as printed — and you may add the 49 remark as a clearly-labelled note.

Conclusion

For part (a), expressing the stated relation as a 3 : 8 ratio between the roots and using $\alpha + \beta = -b/a$ with $\alpha\beta = q/a$ yields the necessary and sufficient condition $121aq = 24b^2$, verified numerically on the roots 3 and 8. For part (b), the printed conditions are mutually inconsistent — a reversal gain is always a multiple of 9, and only 45, 0 and −45 arise from digit-product 36 — so as printed there is no such number; the evidently intended variant with gain 45 has the answer 49.

Necessity, sufficiency and a word of caution

The condition found in part (a), $121aq = 24b^2$, is both necessary and sufficient: it was derived from the stated root relation using only the sum and product of the roots, and conversely any

quadratic satisfying it has roots whose ratio is 3 : 8 (in some order), as the reader can confirm by setting $\alpha + \beta = -b/a$ and $\alpha\beta = q/a$ with the ratio inserted. One caution about signs is in order. The derivation squares nothing and divides only by quantities that are non-zero in a genuine quadratic ($a \neq 0$ by definition, and the ratio step assumes the roots are not both zero), so no extraneous condition has crept in and none has been lost. Conditions of this type — a single polynomial relation among the coefficients — are the standard currency of 'find the condition that the roots…' problems throughout this unit, and the method used here, translate the root condition into the ratio or difference of the roots and then eliminate the roots using sum and product, is the template for all of them.

Question No. 5 [20]

Question No. 5: (a) Differentiate between proper and improper rational fractions with examples. Let $p(\gamma) = \gamma^4 + 2\gamma^3 + C\gamma^2 + 3$. If $p(5) = C^2$ find POSSIBLE values of C. (b) Resolve into partial fractions: $(3 + y^2)/(y^2 - 12y + 32)$.

Introduction

A rational fraction is a quotient of two polynomials. Whether it is called proper or improper depends on a simple comparison of degrees, and the distinction matters because the two kinds are handled differently: proper fractions can be split directly into partial fractions, while improper ones must first be divided out. Part (a) asks for that distinction with examples and then applies polynomial evaluation to find a constant C; part (b) is itself a perfect illustration of the distinction, because the fraction given there is improper and refuses to be resolved until the division step has been done.

Part (a) — Proper and improper rational fractions

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Let the fraction be f(x)/g(x) with g(x) not the zero polynomial. The fraction is proper if the degree of the numerator f is strictly less than the degree of the denominator g, and improper if the degree of the numerator is greater than or equal to the degree of the denominator. Examples of proper fractions: $(2x + 1)/(x^2 + 3x + 5)$ (degrees 1 and 2), $5/(x^3 - 1)$ (degrees 0 and 3), and $(x^2 - 4)/(x^4 + 1)$ (degrees 2 and 4). Examples of improper fractions: $(x^3 + 2)/(x + 1)$ (degrees 3 and 1), $(x^2 + 3)/(x^2 - 12x + 32)$ (degrees 2 and 2 — equality already makes it improper), and $(4x^4)/(x^2 + 9)$ (degrees 4 and 2).

Feature	Proper rational fraction	Improper rational fraction
Degree comparison	deg(numerator) < deg(denominator)	deg(numerator) ≥ deg(denominator)
Example	$(2x + 1)/(x^2 + 3x + 5)$	$(x^2 + 3)/(x^2 - 12x + 32)$
Behaviour as $x \to \infty$	Tends to 0	Tends to a constant or grows like a polynomial
Partial fractions	Can be resolved directly	Divide first: polynomial + proper remainder, then resolve the remainder
Analogy	Like 3/7, a 'top-light' fraction	Like 9/7 = 1 + 2/7, a mixed form after division

The analogy in the last row is the one to remember from school arithmetic: just as the improper number-fraction 9/7 is rewritten as 1 + 2/7 before further work, every improper rational fraction is rewritten, by polynomial long division, as a polynomial plus a proper fraction, and only the proper part is then resolved into partial fractions. Part (b) below carries out exactly this programme.

Now the constant. Evaluating $p(\gamma) = \gamma^4 + 2\gamma^3 + C\gamma^2 + 3$ at $\gamma = 5$ gives $p(5) = 5^4 + 2\cdot5^3 + C\cdot5^2 + 3 = 625 + 250 + 25C + 3 = 878 + 25C$. The condition $p(5) = C^2$ therefore becomes $C^2 = 878 + 25C$, or, collecting everything on one side,

$$C^2 - 25C - 878 = 0.$$

This does not factorise over the integers (no pair of integers multiplies to -878 and adds to -25), so the quadratic formula is the right tool: $C = [25 \pm \sqrt{(625 + 3512)}]/2 = (25 \pm \sqrt{4137})/2$. The discriminant 4137 lies between $64^2 = 4096$ and $65^2 = 4225$ and is not a perfect square, so the two possible values are irrational and are best left in exact form, with decimal approximations alongside:

$$C = (25 + \sqrt{4137})/2 \approx 44.66 \quad \text{or} \quad C = (25 - \sqrt{4137})/2 \approx -19.66.$$

Both values genuinely work, which is why the booklet says 'POSSIBLE values' in the plural. Check the first: $C \approx 44.66$ gives $C^2 \approx 1994.5$ and $878 + 25C \approx 878 + 1116.5 = 1994.5$. Check the second: $C \approx -19.66$ gives $C^2 \approx 386.5$ and $878 + 25C \approx 878 - 491.5 = 386.5$. A common error here is to 'solve' $C^2 = 878 + 25C$ by taking square roots term by term; the equation is a quadratic in C and must be treated as one — move all terms to one side first.

Part (b) — Resolving $(3 + y^2)/(y^2 - 12y + 32)$ into partial fractions

First decide the type: numerator and denominator both have degree 2, so by part (a) the fraction is improper and we divide before resolving. The denominator factorises as $y^2 - 12y + 32 = (y - 4)(y - 8)$. Dividing the numerator by the denominator: $y^2 + 3 = 1 \times (y^2 - 12y + 32) + (12y - 29)$, so

$$(y^2 + 3)/(y^2 - 12y + 32) = 1 + (12y - 29)/[(y - 4)(y - 8)], \quad y \neq 4, 8.$$

The remainder fraction is proper (degrees 1 and 2), so we may write $(12y - 29)/[(y - 4)(y - 8)] = A/(y - 4) + B/(y - 8)$. Clearing denominators gives the identity $12y - 29 = A(y - 8) + B(y - 4)$,

true for all y. Substitute the two convenient values. With y = 4: 48 − 29 = 19 = A(−4), so A = −19/4. With y = 8: 96 − 29 = 67 = B(4), so B = 67/4. Therefore the resolution is

$$(3 + y^2)/(y^2 - 12y + 32) = 1 - 19/[4(y - 4)] + 67/[4(y - 8)].$$

Verification: recombining the two partial fractions over the common denominator 4(y − 4)(y − 8), the numerator is −19(y − 8) + 67(y − 4) = −19y + 152 + 67y − 268 = 48y − 116 = 4(12y − 29), which returns exactly (12y − 29)/[(y − 4)(y − 8)]; adding the quotient 1 then restores (y² + 3)/(y² − 12y + 32). A spot-check with a number, y = 0, is even quicker: the original fraction is 3/32 = 0.09375, and the resolved form gives 1 + 19/16 − 67/32 = 1 + 1.1875 − 2.09375 = 0.09375. Agreement confirmed.

Conclusion

A rational fraction is proper exactly when the numerator's degree is the smaller one; improper fractions are first divided into a polynomial plus a proper remainder. The condition p(5) = C² reduces to C² − 25C − 878 = 0, with the two possible values C = (25 ± √4137)/2 ≈ 44.66 and ≈ −19.66. And part (b) demonstrates the whole theory in one line of work: (3 + y²)/(y² − 12y + 32) = 1 − 19/[4(y − 4)] + 67/[4(y − 8)], valid for y ≠ 4, 8, and verified by recombining the fractions.

A note on the cover-up method and its limits

The substitutions y = 4 and y = 8 used in part (b) are often called the cover-up method: to find the coefficient of 1/(y − 4) one 'covers up' the factor (y − 4) in the original proper fraction and evaluates what remains at y = 4 — here (12y − 29)/(y − 8) at y = 4 gives 19/(−4) = −19/4, exactly our A. The method works cleanly because every factor of the denominator is linear and distinct. When a denominator contains a repeated factor such as (y − 4)², or an irreducible quadratic such as y² + 1, the assumed form of the resolution changes — extra terms with the repeated factor to both powers, or a linear numerator over the quadratic, are required. None of

that arises in the present question, whose denominator splits into two distinct linear factors, but recognising which case one is in, before writing A and B down, is the mark of a reliable partial-fractions worker.

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